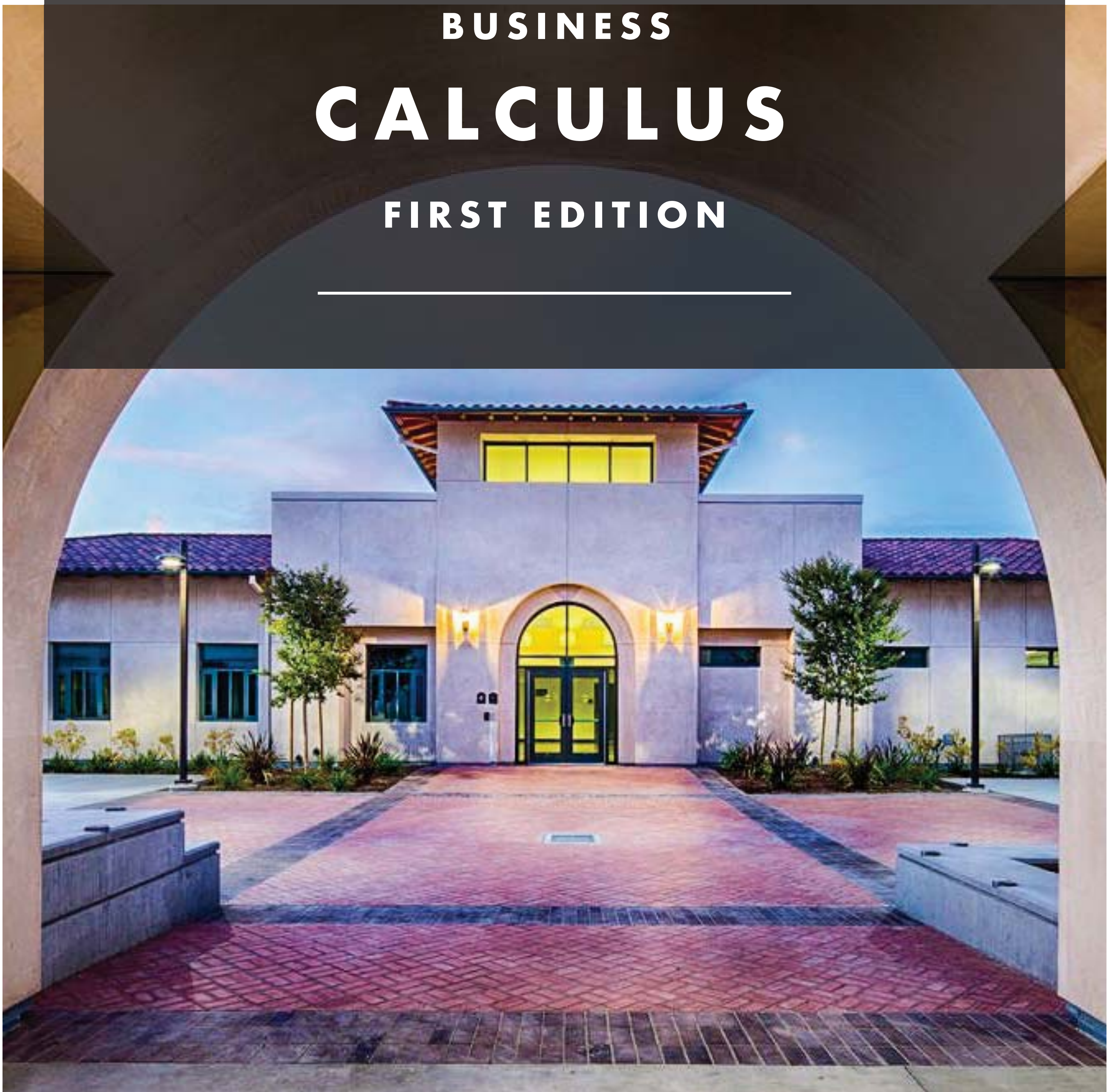
BUSINESS

CALCULUS

FIRST EDITION



LBCC CUSTOM EDIT

S. NGUYEN, R. KEMP

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Introduction

A Preview of Calculus

Calculus was first developed more than three hundred years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons.

Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion.

Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic and social phenomena and to describe and solve problems in those areas.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields.

About this book

Chapter 1: Derivatives and their Applications.

Chapter 2: Extended Applications of Derivative.

Chapter 3: Logarithmic and Exponential Functions.

Chapter 4: Calculus of Several Variables

Chapter 5: Integration Techniques

How is Business Calculus Different?

Students who plan to go into science, engineering, or mathematics take a year-long sequence of classes that cover many of the same topics as we do in our one-quarter or one-semester course.

Here are some of the differences:

No trigonometry

We will not be using trigonometry at all in this course. The scientists and engineers need trigonometry frequently, and so a great deal of the engineering calculus course is devoted to trigonometric functions and the situations they can model.

The applications are different

The scientists and engineers learn how to apply calculus to physics problems, such as work. They do a lot of geometric applications, like finding minimum distances, volumes of revolution, or arclengths.

In this class, we will do only a few of these (distance/velocity problems, areas between curves). On the other hand, we will learn to apply calculus in some economic and business settings, like maximizing profit or minimizing average cost, finding elasticity of demand, or finding the present value of a continuous income stream.

These are applications that are seldom seen in a course for engineers.

Fewer theorems, no proofs

The focus of this course is applications rather than theory. In this course, we will use the results of some theorems, but we won't prove any of them. When you finish this course, you should be able to solve many kinds of problems using calculus, but you won't be prepared to go on to higher mathematics.

Business Calculus

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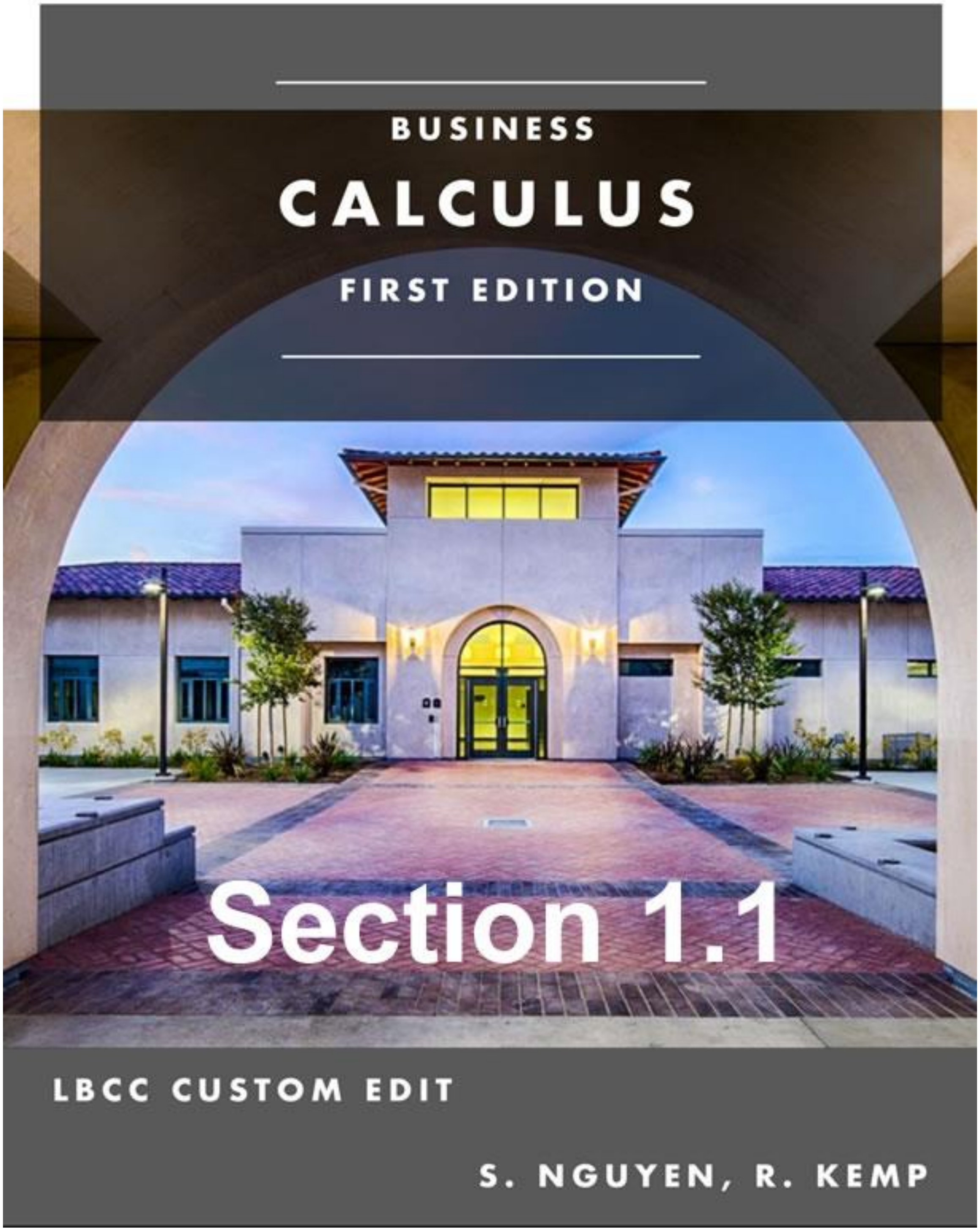
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Section 1.1
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1.1 - LIMITS AND CONTINUITY

The limit of a function describes the behavior of the function when the variable is near, **but does not equal**, a specified number (Fig. 1).

If the values of f(x) get closer and closer, as close as we want, to one number, L, as we choose values of x that get very close to (but not equal to) a number c, then

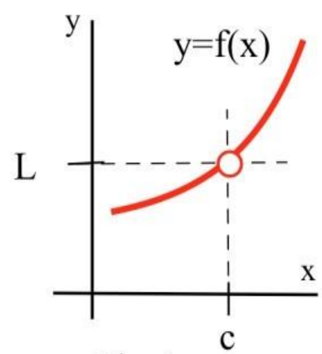


Fig. 1

We say **"the limit of $f(x)$, as x approaches c, is L"** and we write

$$\lim_{x \to c} f(x) = L$$

(The symbol " → " means "approaches" or "gets very close to.")

Please understand the difference between the value of a function at a point and the limit of the function at that point.

- ($f(c)$) is a single number that describes the behavior (value) of ($f(x)$),

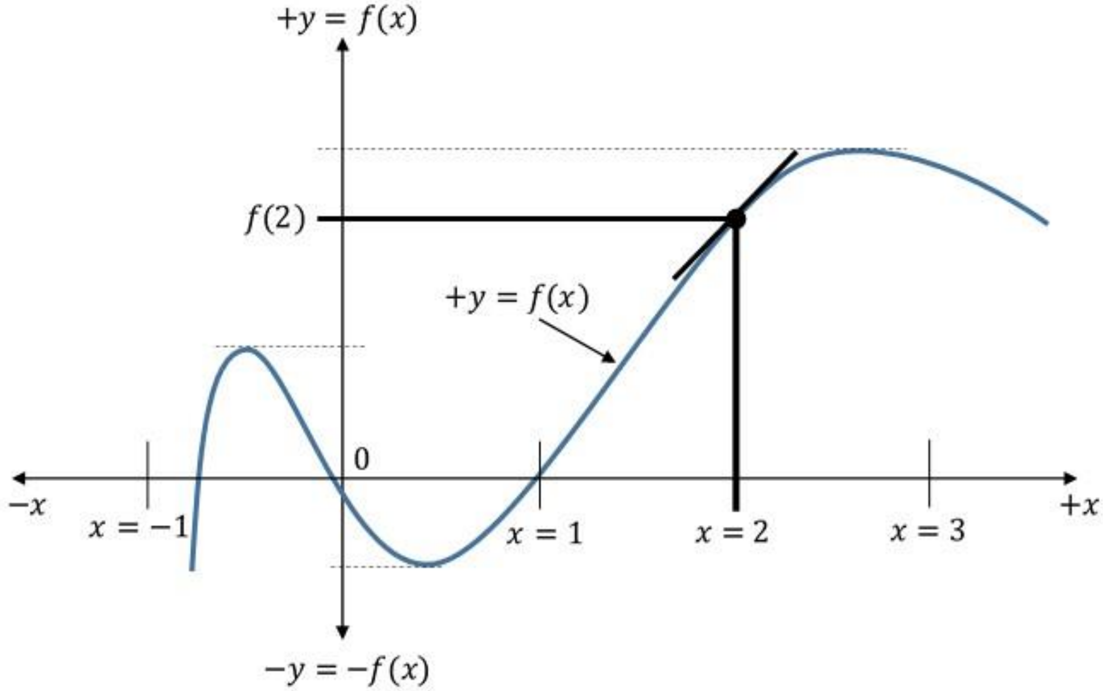
  AT the point x = c.

- $\lim_{x \to c} f(x)$ is a single number that describes the behavior of ($f(x)$).

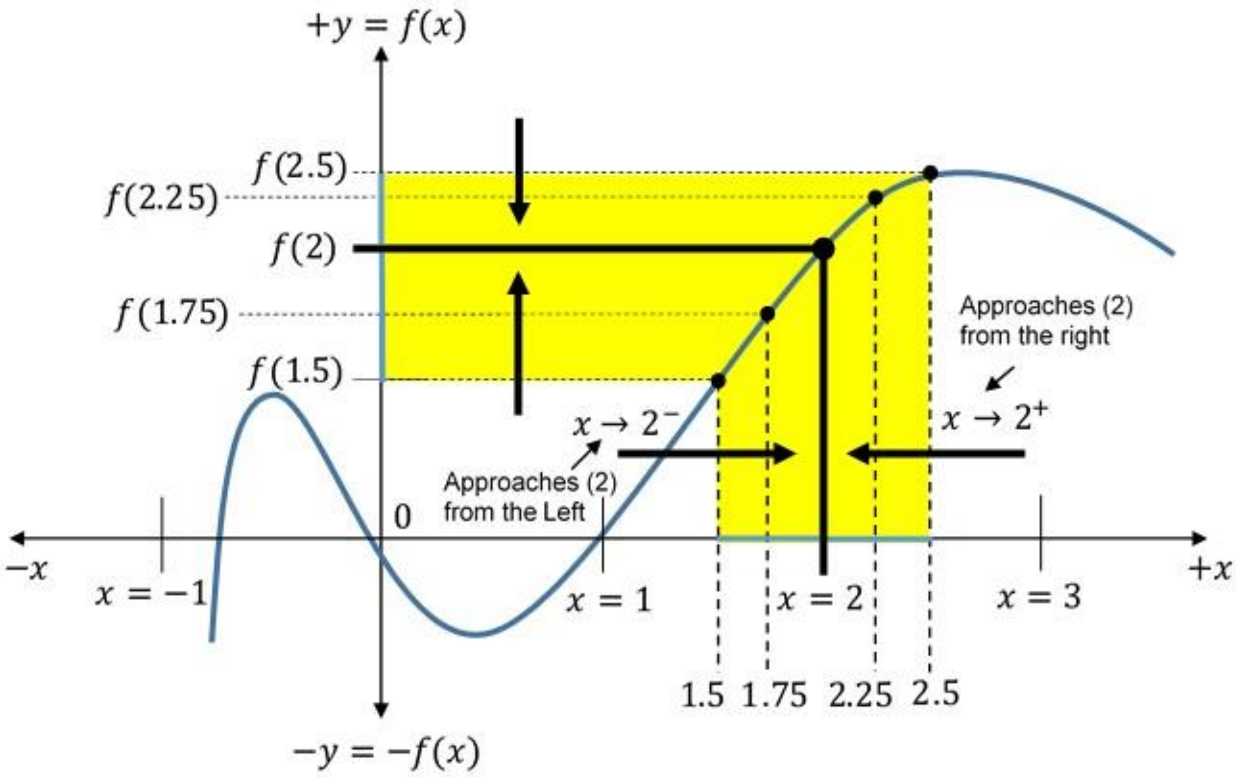
  NEAR, BUT NOT AT, the point x = c.

If we have a graph of the function near (x → c), then it means that the values of (x) become randomly close to the value of (c), but never actually reach the value (x = c); the math notation is: $\lim_{x \to c} f(x)$.

In the graph of the function near (x = c = 2), below; the limit is determined to be the value of ($f(2)$), in the limit as (x) approaches (2): $\lim_{x \to 2} f(x) = f(2)$.



However, the value (x = 2) can be approached, from either the left-hand (**$x \to 2^-$**) side, or the right-hand (**$x \to 2^+$**) side; see image below. More to this in a few pages.



Example Problem #1

Use the following graph to determine the following limits:

a) $\lim_{x \to 1} f(x)$

b) $\lim_{x \to 2} f(x)$

c) $\lim_{x \to 3} f(x)$

d) $\lim_{x \to 4} f(x)$

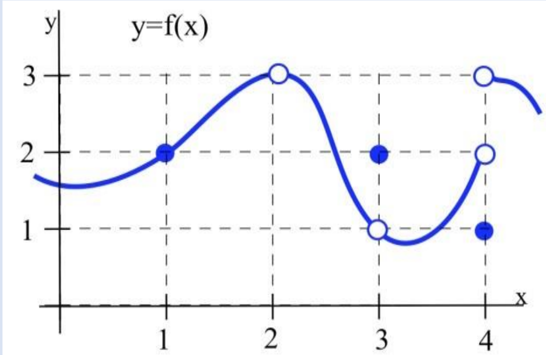


Fig. 2

Solution:

(a) When x is very close to 1, the values of f(x) are very close to y = 2. In this example, it happens that *f* (1) = 2, but that is irrelevant for the limit. The only thing that matters is what happens for x close to 1 but x ≠ 1, so $\lim_{x \to 1} f(x) = 2$.

(b) *f* (2) is undefined, but we only care about the behavior of *f* (x) for x close to 2 and not equal to 2. When x is close to 2, the values of *f* (x) are close to 3. If we restrict x close enough to 2, the values of y will be as close to 3 as we want, so $\lim_{x \to 2} f(x) = 3$.

(c) When x is close to 3 (or as x approaches the value 3), the values of f(x) are close to 1 (or approach the value 1), so $\lim_{x \to 3} f(x) = 1$. For this limit, it is completely irrelevant that *f* (3) = 2, we only care about what happens to *f*(x) for x close to and not equal to 3.

(d) This one is harder and we need to be careful. When x is close to 4 and slightly **less than** 4 (x is just to the left of 4 on the x–axis), then the values of *f* (x) are close to 2. But, if x is close to 4 and slightly **larger than** 4 then the values of *f* (x) are close to 3. If we only know that x is very close to 4, then we cannot say whether y = *f* (x) will be close to 2 or close to 3 — it depends on whether x is on the right or the left side of 4. In this situation, the *f* (x) values are not close to a single number so we say $\lim_{x \to 4} f(x)$ **does not exist**. It is irrelevant that *f* (4) = 1. The limit, as x approaches 4, would still be undefined if *f* (4) was 3 or 2 or anything else.

Example Problem #2

Let us now explore limits using tables, algebra and graphs.

Find:

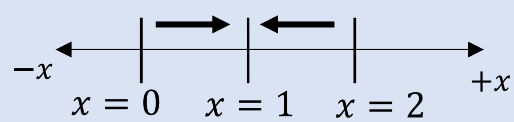
$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1}$$

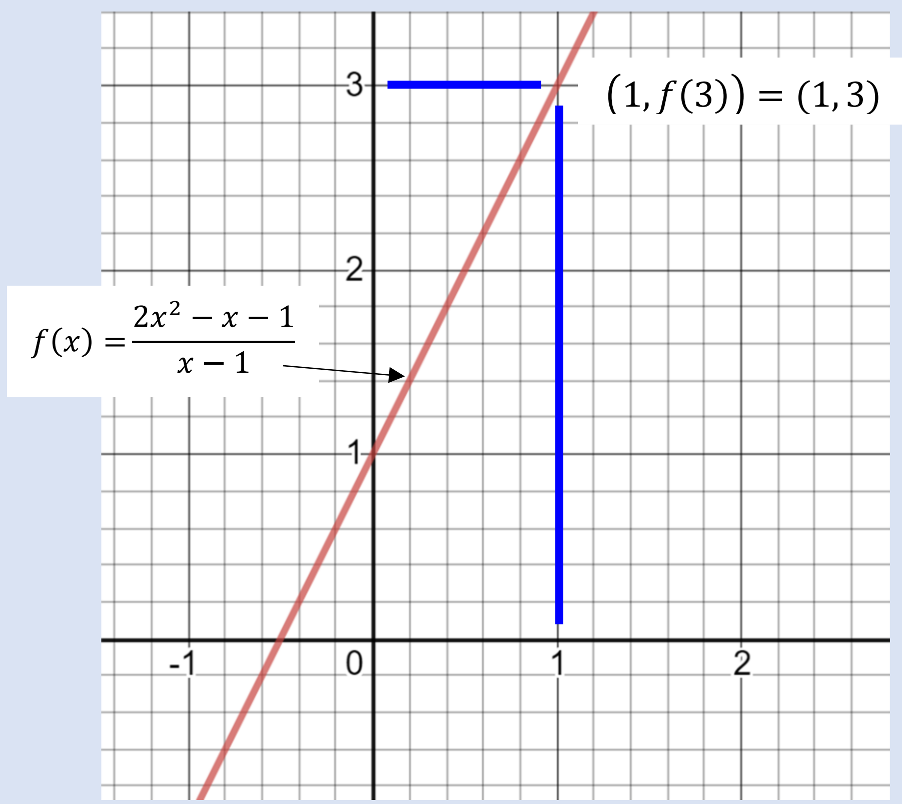
You might try to evaluate:

$$f(x) = \frac{2x^2 - x - 1}{x - 1} \qquad at \quad x = 1$$

but *f* (x) is not defined at x = 1.

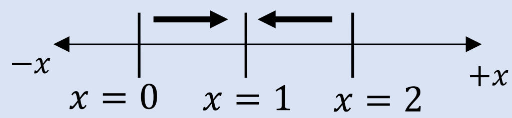
It is tempting, **but wrong**, to conclude that this function does not have a limit as x approaches 1.





Example Problem #2

Let us now explore limits using tables.



Solution using tables:

Trying some "test" values for x which get closer and closer to 1 from both left and right, we get

$\lim_{x\to 1^-} \frac{2x^2 - x - 1}{x - 1}$	
x	$f(x) = \frac{2x^2 - x - 1}{x - 1}$
0.9	2.82
0.9998	2.9996
0.99994	2.999988
0.9999999	2.9999998

$\lim_{x\to 1^+} \frac{2x^2 - x - 1}{x - 1}$	
x	$f(x) = \frac{2x^2 - x - 1}{x - 1}$
1.1	3.2
1.003	3.006
1.0001	3.0002
1.000007	3.000014

The function is not defined at $(x = 1)$, but when (x) is close to (1), the limit values of $f(x)$ as you approach (1) from the right and left, are getting close to 3.

We can get *f* (x) as close to 3 as we want by taking x very close to 1, so, we conclude that the limit of the function exists!

Since the limiting value approaching from the right $(x \to 1^+)$ equals to the same limiting value approaching from the left $(x \to 1^-)$, then the limit of the function exists!

$$\lim_{x\to 1^-} \frac{2x^2 - x - 1}{x - 1} = \lim_{x\to 1^+} \frac{2x^2 - x - 1}{x - 1}$$

And therefore, we conclude that the limit of the function exists!

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{x - 1} = 3$$

Example Problem #2 – Cont'd

Solution using algebra:

We could have found the same result by noting that,

$$f(x) \quad = \quad \frac{2x^2 - x - 1}{x - 1} \quad = \quad \frac{(2x+1)(x-1)}{(x-1)} \quad = \quad 2x + 1$$

(We are not allowed to divide by 0 but, if x≠1, then (x–1) ≠ 0, so it is valid to divide the numerator and denominator by the factor (x–1)).

Remember, the "x→1" part of the limit means that x is close to1, but **not equal to 1**, so our division step is valid and;

$$\lim_{x \to 1} f(x) \quad = \quad \lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} \quad = \quad \lim_{x \to 1} \frac{(2x+1)(x-1)}{x - 1}$$

$$\lim_{x \to 1} f(x) \quad = \quad \lim_{x \to 1} (2x \ + \ 1) \ = \ 3$$

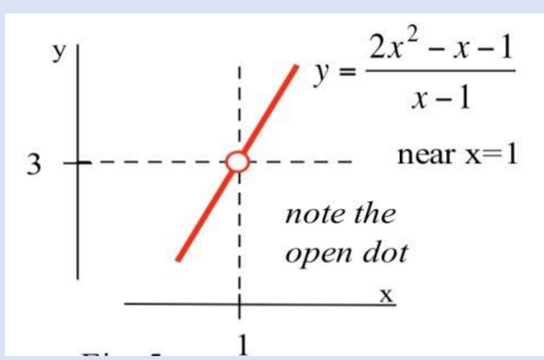
Note that we only substitute x with 1 when the denominator no longer equals 0!

Solution using a graph:

We can graph $\quad f(x) = \frac{2x^2 - x - 1.}{x - 1}$

Notice that whenever x is close to 1, the values of f(x) are close to 3.

The function ($f(x)$) is not defined at x = 1, so the graph has a hole where x = 1, but when we discuss a limit, we only care about what f(x) is doing for x close to but **not equal to** the value x = 1.



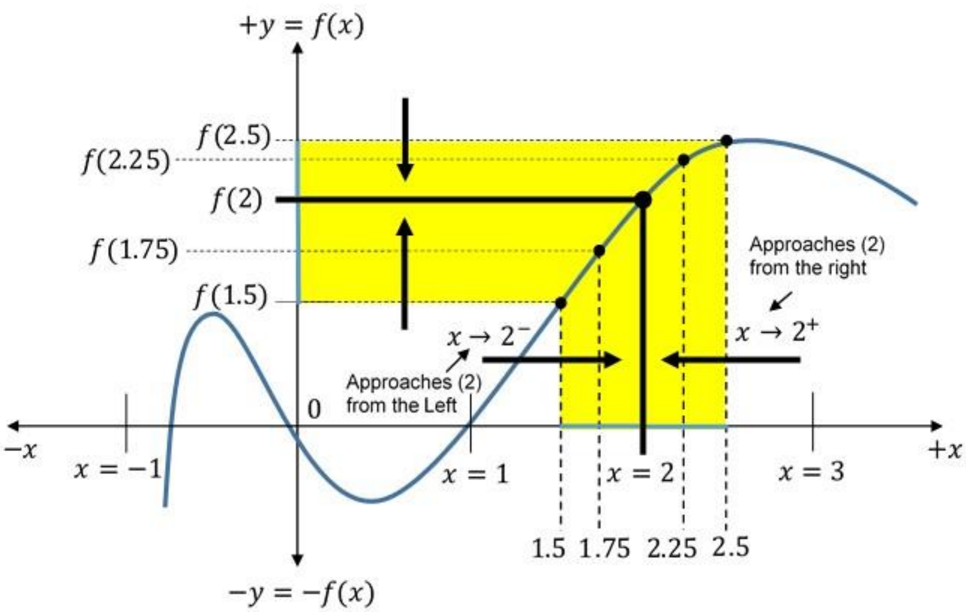
One Sided Limits

Sometimes, what happens to us at a place depends on the direction we use to approach that place.

If we approach Niagara Falls from the upstream side, then we will be 182 feet higher than the downstream side; and we would have different worries than if we approached from the downstream side.

https://en.wikipedia.org/wiki/Niagara_Falls#/media/File:3Falls_Niagara.jpg

Similarly, the values of a function near a point may depend on the direction we use to approach that point. In the graph of the function near (c = 2), below; the limit is determined to be the value of ($f(2)$), if the limit as (x) approaches the value (2); either from the left-hand (**x → 2**⁻) side: [$\lim_{x \to 2^-} f(x) = f(2)$], or approaches from the right-hand side (**x → 2**⁺): [$\lim_{x \to 2^+} f(x) = f(2)$].



Definition of Left and Right Limits:

The **left limit** as (x) approaches (c) of $(f(x))$ is (L), if the values of $(f(x))$ get as close to (L) as we want when (x) is very close to and **left of** (c), x **< c**:

$$\lim_{x \to c^-} f(x)$$

The **right** as (x) approaches (c) of $(f(x))$ is (L), if the values of $(f(x))$ get as close to (L) as we want when (x) is very close to and **right of** (c), x **> c**:

$$\lim_{x \to c^+} f(x)$$

If both the left limit and the right limit equal to the same value, then the limit (also sometimes called the two-sided limit) will **exist.**

$$If \lim_{x \to c^-} f(x) \quad = \quad \lim_{x \to c^+} f(x) = L \text{ , then}$$

$$\lim_{x \to c} f(x) \quad = \quad L$$

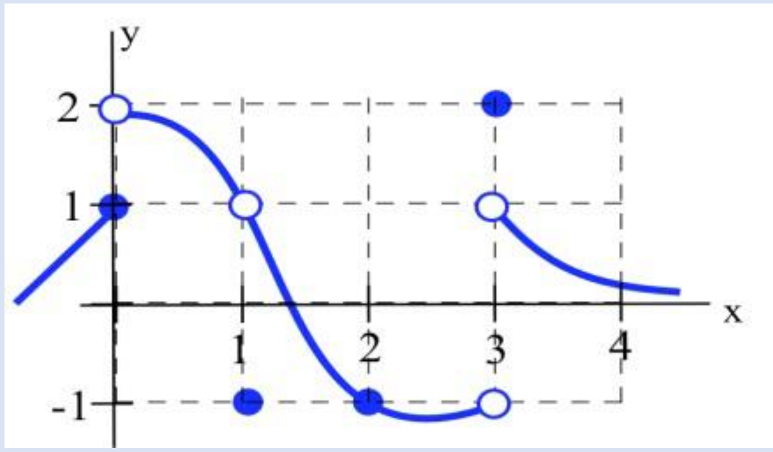
If both the left limit and the right limit do not equal to the same value, then the limit (also sometimes called the two-sided limit) does not exist (D.N.E).

$$If \lim_{x \to c^-} f(x) \quad \neq \quad \lim_{x \to c^+} f(x), \text{ then}$$

$$\lim_{x \to c} f(x) \quad = \quad Does\ Not\ Exist\ (D.N.E)$$

Example Problem #3

Evaluate the one-sided limits of the function *f* (x) at x = 0 and x = 1.



Solution:

As x approaches 0 from the left, the value of the function is getting closer to 1, so:

$$\lim_{x \to 0^-} f(x) = 1.$$

As x approaches 0 from the right, the value of the function is getting closer to 2, so:

$$\lim_{x \to 0^+} f(x) = 2.$$

Note that since the limit from the left and the limit from the right are different, the general limit ($\lim_{x \to 0} f(x)$), does not exist (D.N.E).

$$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0} f(x) = D.N.E$$

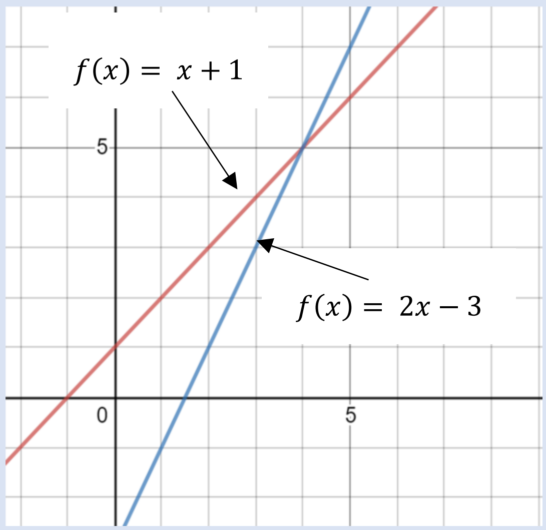
As x approaches 1 from either side, the value of the function is getting closer to 1, so:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1$$

Example Problem #4

Given the function $f(x) = \begin{cases} x+1, & x<2 \\ 2x-3, & x \geq 2 \end{cases}$, find

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$



Solution:

(a) Since we are finding the left sided limit, the x values must be less than 2, so we need to substitute x-values less than 2, hence we need to use the first part of the definition of f(x)

$$\lim_{x \to 2^-} f(x) = 2 + 1 = 3$$

(b) Since we are finding the right sided limit, the x values must be greater than 2, so we need to substitute x-values greater than 2, hence we need to use the second part of the definition of f(x)

$$\lim_{x \to 2^+} f(x) = 2(2) - 3 = 1$$

Example Problem #4 – Cont'd

(c) Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\lim_{x \to 2} f(x) = D.N.E$ (does not exist).

Going back to example 1d), we saw that the $\lim_{x \to 4} f(x) = D.N.E$ (does not exist).

However, we do have one-sided limits. $\lim_{x \to 4^-} f(x) = 2$ and $\lim_{x \to 4^+} f(x) = 3$.

The limit at point exists, only if the limit from the right equals the limit form the left.

Continuity

A function that doesn't have any breaks or jumps in it, i.e. we can draw its graph without lifting the pen from the paper is called **continuous**. More formally,

Definition of Continuity at a Point

A function **f is continuous at a point x = a** if and only if: $\lim_{x \to a} f(x) = \boldsymbol{f(a)}$

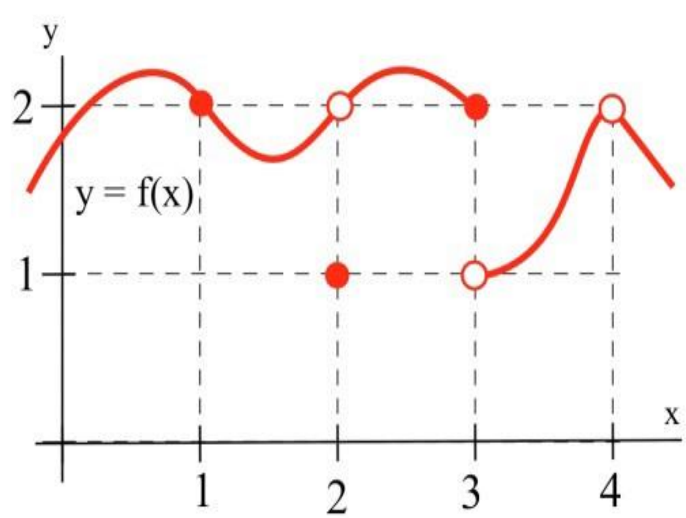
In other words,

1) the limit when (x → a) exists, $\lim_{x \to a} f(x) = L,$
2) the function is defined at (x = a), $f(a) = L$
3) they equal the same number.

A function is said to be continuous if it is continuous at ever point in its domain.

The graph below illustrates some of the different ways a function can behave at and near a point, and the table contains some numerical information about the function and its behavior.

Based on the information in the table, we can conclude that (f) is continuous at (x = 1), since ($\lim_{x \to 1} f(x) = f(1) = 2$)



We can also conclude from the information in the table that (f) is not continuous at 2 or 3 or 4, because

$$\lim_{x\to 2} f(x) \neq f(2), \quad \lim_{x\to 3} f(x) \neq f(3), \quad \lim_{x\to 4} f(x) \neq f(4).$$

a	$f(a)$	$\lim_{x\to a} f(x)$	Continuous Yes/No
1	2	2	Yes
2	1	2	No
3	2	does not exist	No
4	undefined	2	No

The behaviors at *x* = 2 and *x* = 4 exhibit a **hole** in the graph, sometimes called a **removable discontinuity**, since the graph could be made continuous by changing the value of a single point.
The behavior at *x* = 3 is called a **jump discontinuity**, since the graph jumps between two values.

So, which functions are continuous? It turns out pretty much every function you've studied is continuous where it is defined: polynomial, radical, rational, exponential, and logarithmic functions are all continuous where they are defined. Moreover, any combination of continuous functions is also continuous.

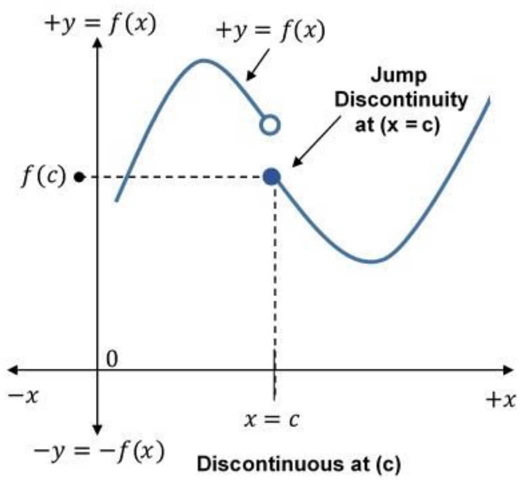
This is helpful, because the definition of continuity says that for a continuous function,

$$\lim_{x\to a} f(x) = f(a)$$

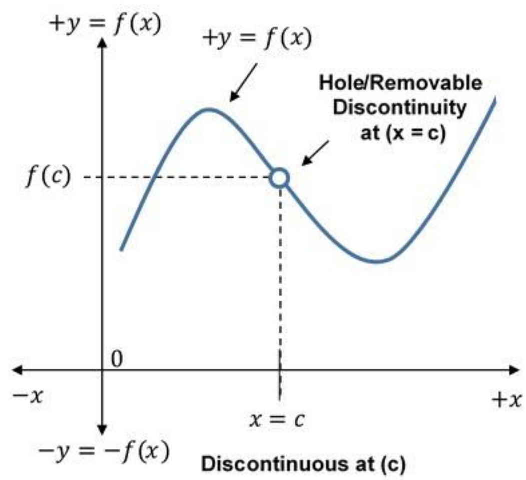
That means for a continuous function, we can find the limit by direct substitution (evaluating the function) if the function is continuous at *a*.

The graphs below illustrate some of the different ways a function can behave at and near a point (x = c).

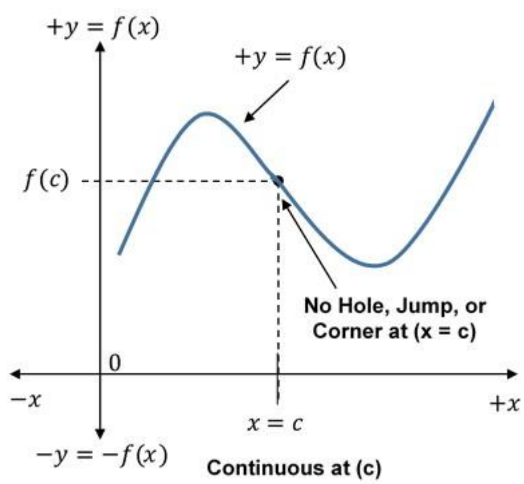
- **Graph (a)**, there is a **"jump" discontinuity** in the function at (x = c).
- **Graph (b)**, there is a **"hole" discontinuity** in the function at (x = c).
- **Graph (c)**, the function **is continuous**; there are no breaks, holes, or jump discontinuities in the function at (x = c).



(a)



(b)



(c)

Example Problem #5

Evaluate using continuity, if possible:

a) $\lim_{x\to 8=2^3} \sqrt[3]{x}$ b) $\lim_{x\to 2} (x^3 - 4x)$ c) $\lim_{x\to 2} \frac{x-4}{x+3}$ d) $\lim_{x\to 2} \frac{1}{x-2}$

Solution:

a) The given function is a root function, and is defined for all values of $(x = 8 = 2^3)$, so we can find the limit, by direct substitution:

$$\lim_{x\to 8} \sqrt[3]{x} = \sqrt[3]{8} = (8)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$$

b) The given function is a polynomial, and is defined for all values of *x*, so we can find the limit by direct substitution:

$$\lim_{x\to 2}(x^3 - 4x) = (2)^3 - 4(2) = = 8 - 8 = 0$$

c) The given function is rational. It is not defined at *x* = -3, but we are taking the limit as *x* approaches 2, and the function is defined at that point, so we can use direct substitution

$$\lim_{x\to 2} \frac{x-4}{x+3} = \frac{2-4}{2+3} = -\frac{2}{5}$$

d) This function is not defined at *x* = 2, and so is not continuous at *x* = 2. We cannot use direct substitution.

$$\lim_{x\to 2} \frac{1}{x-2} = \frac{1}{2-2} = \frac{1}{0} = Undefined$$

Example Problem #6

Given the function $f(x) = \begin{cases} x+1, & x<2 \\ 2x-1, & x \geq 2 \end{cases}$, determine if (f) is continuous.

Solution

The only value of x at which f could be discontinuous would be x = 2, so let's verify if the definition of continuity holds at x = 2.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x+1) = 2+1 = 3$$

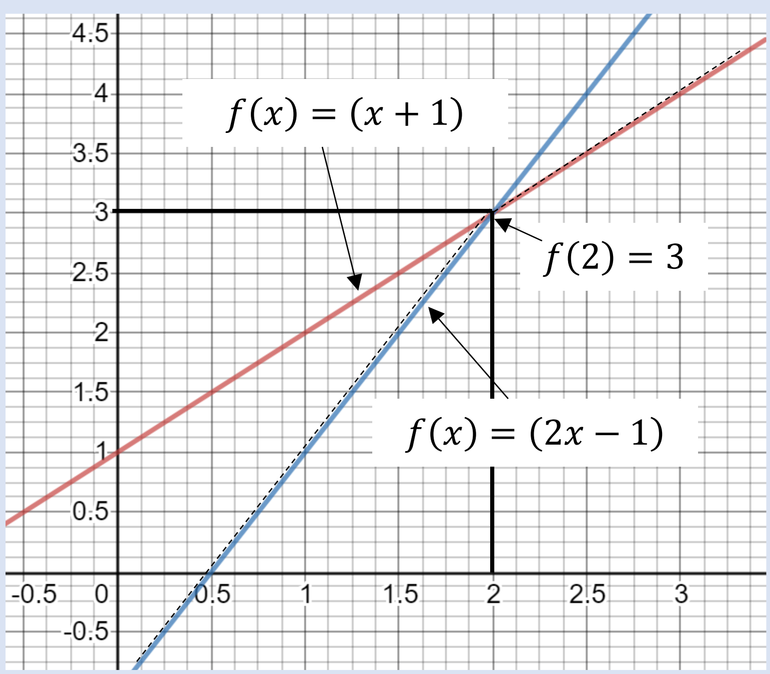
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(2x-1) = 2(2)-1 = 3$$

$$f\,(2) = 2(2) - 1 = 3$$

Since the left-hand limit equals to the right-hand limit, the limit:

$$\lim_{x\to 2} f(x) = 3 \quad \text{and} \quad f(2) = 3,$$

so, the function is continuous at x = 2, based on the definition of continuity.



Summary of Limit Laws

The rules and laws that govern the mathematics of "Limit" functions are given by the following:

1. $\lim\limits_{x \to c} k = k$

2. $\lim\limits_{x \to c} x^n = c^n$

3. $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$

If both the limits exist

$\lim\limits_{x \to c} f(x) = F$ $\qquad$ $\lim\limits_{x \to c} g(x) = G$

4. $\lim\limits_{x \to c} [f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x) = F + G$

5. $\lim\limits_{x \to c} [f(x) - g(x)] = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x) = F - G$

6. $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \left[\lim\limits_{x \to c} f(x)\right] \cdot \left[\lim\limits_{x \to c} g(x)\right] = F \cdot G$

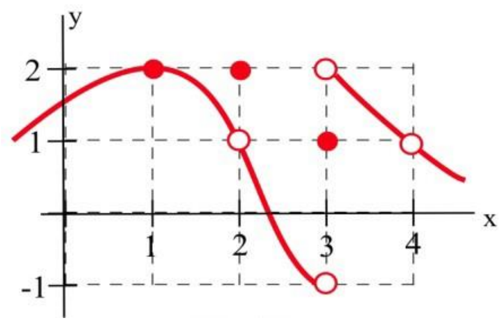
7. $\lim\limits_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{F}{G}$ $\qquad$ $if \quad \lim\limits_{x \to c} g(x) \neq 0$

1.1 - EXERCISES

1. Use the graph to determine the following limits.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $\lim_{x \to 4} f(x)$



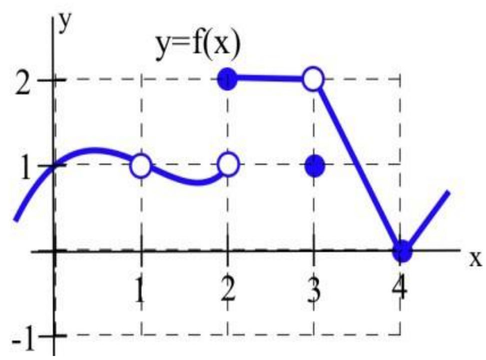
2. Use the graph to determine the following one-sided limits.

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$

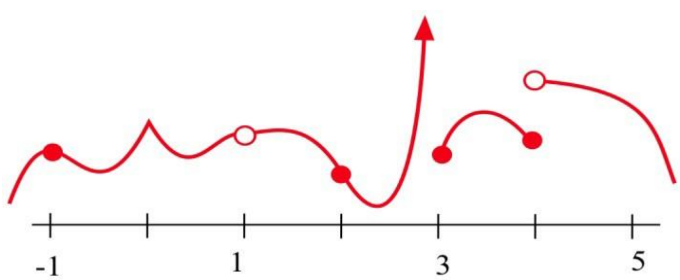
(c) $\lim_{x \to 2^-} f(x)$ (d) $\lim_{x \to 2^+} f(x)$

(e) $\lim_{x \to 3^-} f(x)$ (f) $\lim_{x \to 3^+} f(x)$

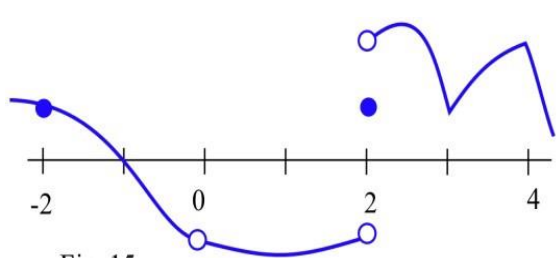
(g) $\lim_{x \to 4^-} f(x)$ (h) $\lim_{x \to 4^+} f(x)$



3. At which points is the function shown discontinuous?



4. At which points is the function shown discontinuous?



5. Find the following limits by using tables:

(a) $\lim_{x\to 2}(3x-7)$

x	3x - 7
1.9	
1.99	
1.999	
1.9999	
2.01	
2.001	
2.0001	

b) $\lim_{x\to 2}\frac{x^2-4}{x-2}$

x	$\frac{x^2-4}{x-2}$
1.9	
1.99	
1.999	
1.9999	
2.01	
2.001	
2.0001	

6. Find the limit by completing a table – Check to see if Limit Exists

** Use a Calculator - Check to five (5) decimal places

$$y = \lim_{x\to 2}\left(\frac{x}{x-2}\right)$$

Left Limit

Approaches (2) from Left

$$y = \lim_{x\to 2^-}\left(\frac{x}{x-2}\right)$$

x	y

Right Limit

Approaches (2) from Right

$$y = \lim_{x\to 2^+}\left(\frac{x}{x-2}\right)$$

x	y

7. Find the following limits by using algebra and substitution.			
a.	$\lim_{x\to 2} \frac{x^2-4}{x-2}$	b.	$\lim_{x\to 1} \frac{x^2-4x+3}{x-1}$
c.	$\lim_{x\to 2} 5$	d.	$\lim_{x\to 6} \sqrt{2x-3}$
e.	$\lim_{h\to 0} \frac{h^2+5h}{h}$	f.	$\lim_{h\to 0} \frac{4x^2h-3xh+5h^2}{h}$
g.	$\lim_{x\to -2} \frac{3x^2+7x+2}{x+2}$	h.	$\lim_{x\to 3} \frac{x^2-9}{x^2+2x-15}$

8. Given the piecewise defined function $f(x) = \begin{cases} 5x-2, x<0 \\ x+1, x\geq 0 \end{cases}$, find

a) $\lim_{x\to 0^-} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0} f(x)$

d) Is f(x) a continuous function? Explain why, or why not

9. Given the piecewise defined function $f(x) = \begin{cases} 7x+2, x<1 \\ x^2+8, x\geq 1 \end{cases}$, find

a) $\lim_{x\to 1^-} f(x)$ b) $\lim_{x\to 1^+} f(x)$ c) $\lim_{x\to 1} f(x)$

d) Is f(x) a continuous function? Explain why, or why not.

10. Given the piecewise defined function $f(x) = \begin{cases} x^3, & x < 0 \\ 5, & 0 \le x < 3 \\ x + 2, & x \ge 3 \end{cases}$ find

a) $\lim_{x \to 0^-} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0} f(x)$

d) $\lim_{x \to 3^-} f(x)$ e) $\lim_{x \to 3^+} f(x)$ f) $\lim_{x \to 3} f(x)$

g) Is f(x) a continuous function? If f is discontinuous, state where and why.

11. Given the function $f(x) = \frac{|x|}{x}$, find

a) $\lim_{x \to 0^-} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0} f(x)$

(Hint: You will need to use tables or a graph for this problem)

12. Given the function $f(x) = \frac{1}{x}$, find

a) $\lim_{x \to 0^-} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0} f(x)$

(Hint: You will need to use tables or a graph for this problem)

Solutions:

1. a) 2, b) 1, c) DNE, d) 1

2. a) 1 , b) 1, c) 1, d) 2, e) 2, f) 2, g) 0, h) 0

3. 1, 3, 4

4. 0, 2

5. a) -1, b) 4.

6. The limit does not exist.

$$y = \lim_{x \to 2} \left(\frac{x}{x - 2} \right) = \text{DNE}$$

$$y = \lim_{x \to 2^-} \left(\frac{x}{x - 2} \right) \neq \lim_{x \to 2^+} \left(\frac{x}{x - 2} \right)$$

Left Limit

Approaches (2) from Left

$$y = \lim_{x \to 2^-} \left(\frac{x}{x - 2} \right) = -\infty$$

x	y
1.9	-19
1.99	-199
1.999	-1999
1.9999	-19999

Right Limit

Approaches (2) from Right

$$y = \lim_{x \to 2^+} \left(\frac{x}{x - 2} \right) = \infty$$

x	y
2.1	21
2.01	201
2.001	2001
2.0001	20001

7. a) 4, b) -2, c) 5, d) 3, e) 5, f) $4x^2 - 3x$, g) -5, h) $\frac{3}{4}$.

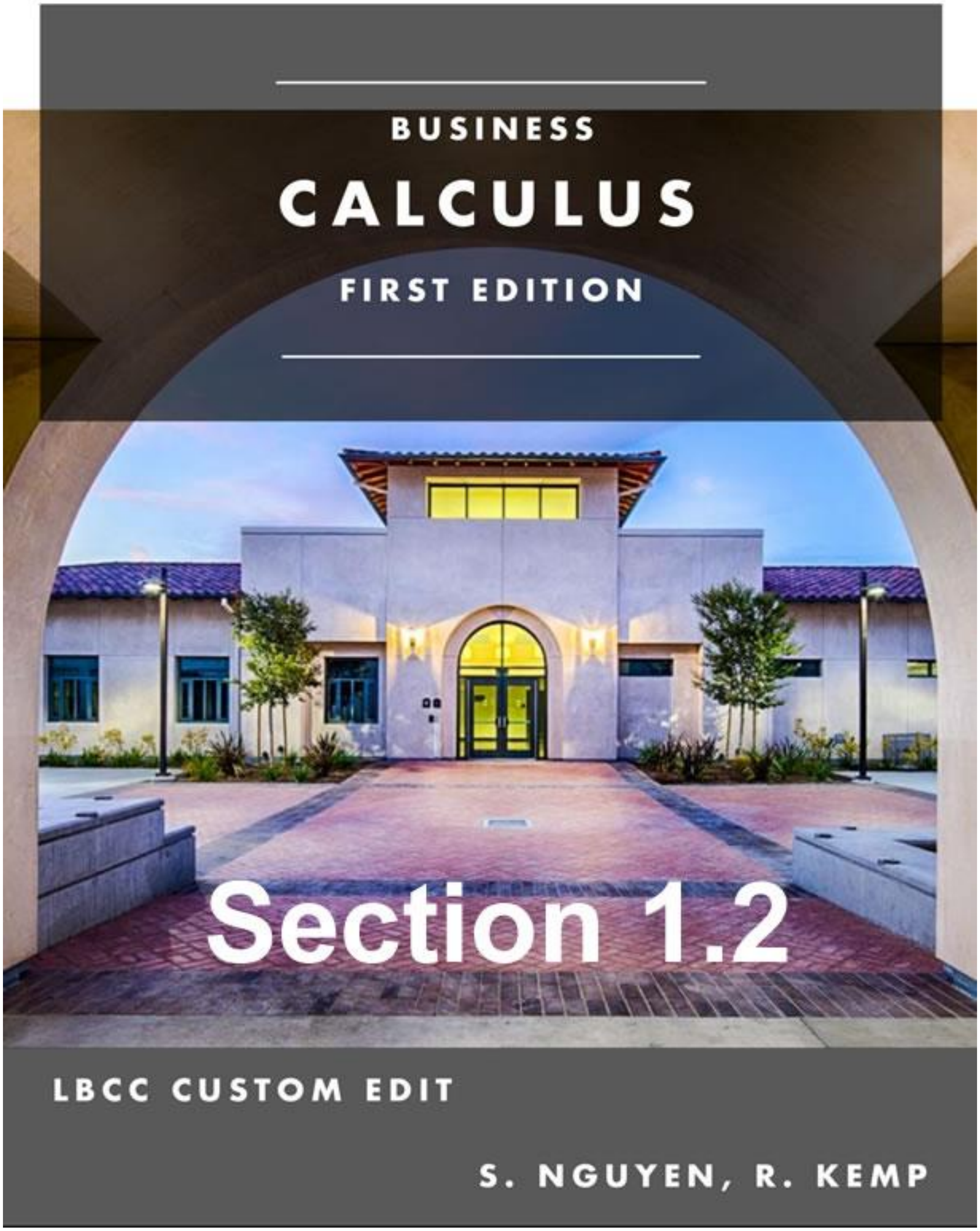
8. a) -2, b) 1, c) DNE, d) no.

9. a) 9, b) 9, c) 9, d) yes.

10. a) 0, b) 5, c) DNE, d) 5, e) 5, f) 5, g) no, discontinuous at x = 0.

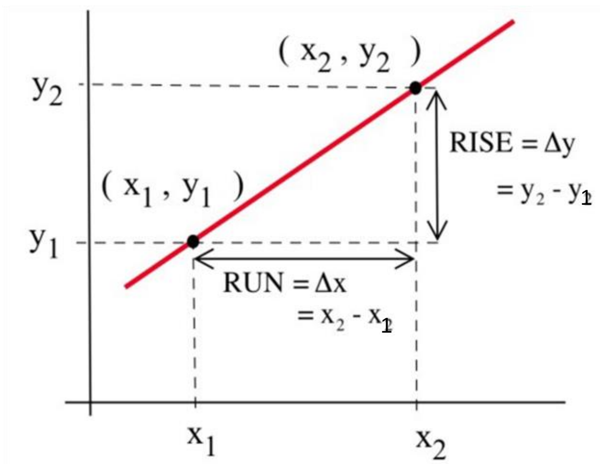
11. a) -1, b) 1, c) DNE

12. a) $-\infty$ or DNE, b) ∞ or DNE, c) DNE.


BUSINESS
CALCULUS
FIRST EDITION
Section 1.2
LBCC CUSTOM EDIT
S. NGUYEN, R. KEMP

1.2 - SLOPE OF THE TANGENT LINE, INSTANTANEOUS RATE OF CHANGE AND DERIVATIVE

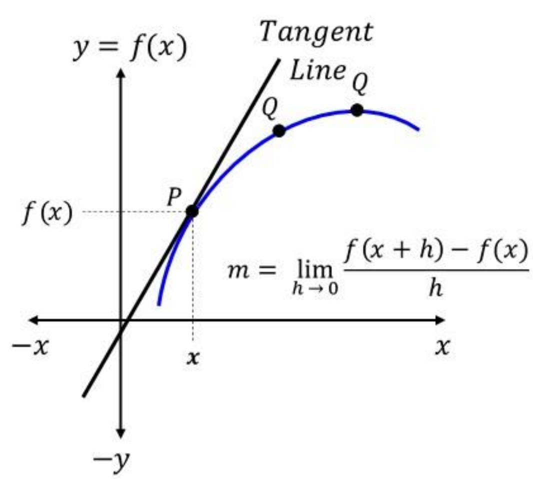
The slope of a line measures how fast a line rises or falls as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. If the line represents the distance traveled over time, for example, then its slope represents the velocity (distance/time).



In the figure, you can remind yourself of how we calculate slope using two points on the line:

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

We would like to be able to get that same sort of information (how fast the curve rises or falls) even if the graph is not a straight line. But what happens if we try to find the slope of a curve, as in the figure below?



We need two points to determine the slope of a line.

How can we find the slope of a curve, at just one point?

The answer, as suggested in the figure above is to find the slope of the tangent line to the curve at that point.

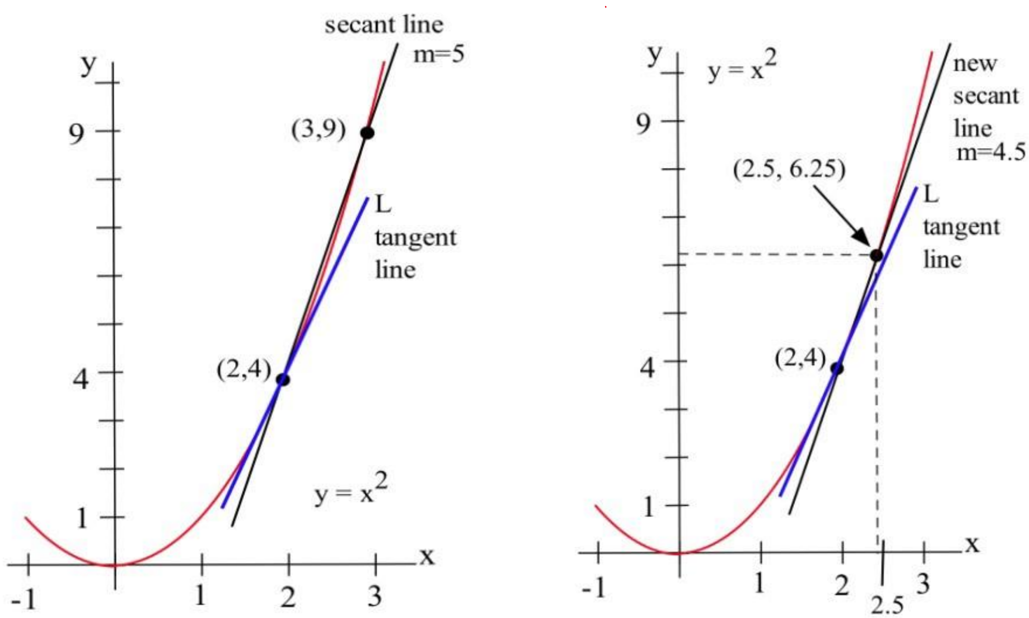
Most of us have an intuitive idea of what a tangent line is. Unfortunately, "tangent line" is hard to define precisely.

Definition: A **secant line** is a line between two points on a curve.

A **tangent line** is a line that touches a curve at one point.

Let's look at the problem of finding the slope of the line L in the graph below which is tangent to the graph of ($f(x) = x^2$) at the point (2,4).

We could estimate the slope of L from the graph, but we won't. Instead, we will use the idea that secant lines over tiny intervals approximate the tangent line.



We can see that the line through (2,4) and (3,9) on the graph of (f) approximates the slope of the tangent line, and we can calculate that slope exactly:

$$m = \frac{9 - 4}{3 - 2} = \frac{\Delta y}{\Delta x} = 5$$

But ($m = 5$) is only an estimate of the slope of the tangent line and not a very good estimate. The distance between the two points we used is too big.

We can get a better estimate by picking a second point on the graph of (f) which is closer to (2, 4).

The point (2, 4) is fixed and it must be one of the points we use.

From the second figure, we can see that the slope of the line through the points (2, 4) and (2.5, 6.25) is a better approximation of the slope of the tangent line at (2,4):

$$m = \frac{\Delta y}{\Delta x} = \frac{6.25 - 4}{2.5 - 2} = 4.5$$

a better estimate, but still an approximation.

We can continue picking points closer and closer to (2,4) on the graph of (f), and then calculating the slopes of the lines through each of these points and the point (2,4):

Points to the left of (2,4)

x	$y = x^2$	Slope of line through (x, y) and (2,4).
1.5	2.25	$m = \frac{\Delta y}{\Delta x} = \frac{2.25 - 4}{1.5 - 2} = 3.5$
1.9	3.61	$m = \frac{\Delta y}{\Delta x} = \frac{3.61 - 4}{1.9 - 2} = 3.9$
1.99	3.9601	$m = \frac{\Delta y}{\Delta x} = \frac{3.9601 - 4}{1.99 - 2} = 3.99$

Points to the right of (2,4)

x	$y = x^2$	Slope of line through (x, y) and (2,4).
3	9	$m = \frac{\Delta y}{\Delta x} = \frac{9 - 4}{3 - 2} = 5$
2.5	6.25	$m = \frac{\Delta y}{\Delta x} = \frac{6.25 - 4}{2.5 - 2} = 4.5$
2.01	4.0401	$m = \frac{\Delta y}{\Delta x} = \frac{4.0401 - 4}{2.01 - 2} = 4.01$

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to (x = 2).

Someone else might have picked other nearby values for x. As the points we pick get closer and closer to the point (2,4) on the graph of ($y = f(x) = x^2$), the slopes of the lines through the points and (2,4) are better approximations of the slope of the tangent line, and these slope values are getting closer and closer to 4.

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near (2,4).

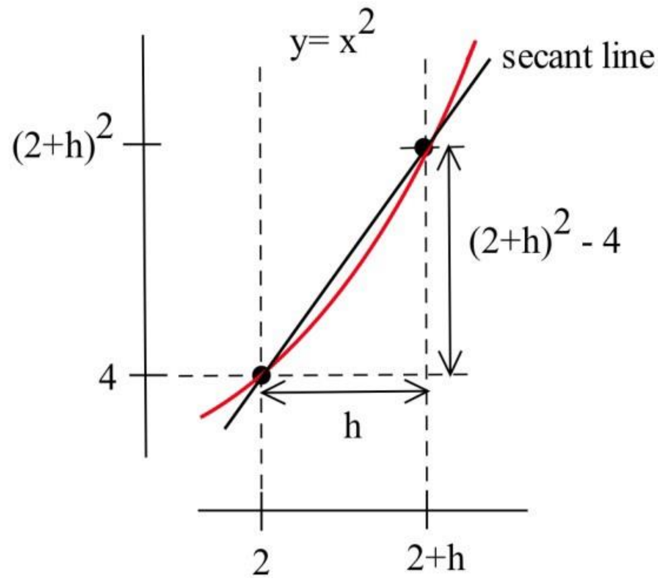
Define ($x = 2 + h$) so that (h) is the increment from (2 to x).

If (h) is small, then ($x = 2 + h$) is close to (2) and the point, on curve:

$$y = f(x) = x^2$$

$(2 + h, f(2 + h))$ = $(2 + h, (2 + h)^2)$ is close to (2, 4).

The slope m of the line through the points (2, 4) and $(2+h,\ (2+h)^2)$ is a good approximation of the slope of the tangent line at the point (2, 4):



$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h$$

The value $(m = 4 + h)$ is the slope of the secant line through the two points (2,4) and $(2+h,\ (2+h)^2)$.

As (h) gets smaller and smaller, i.e. closer and closer to 0, this slope approaches the slope of the tangent line to the graph of (f) at (2,4).

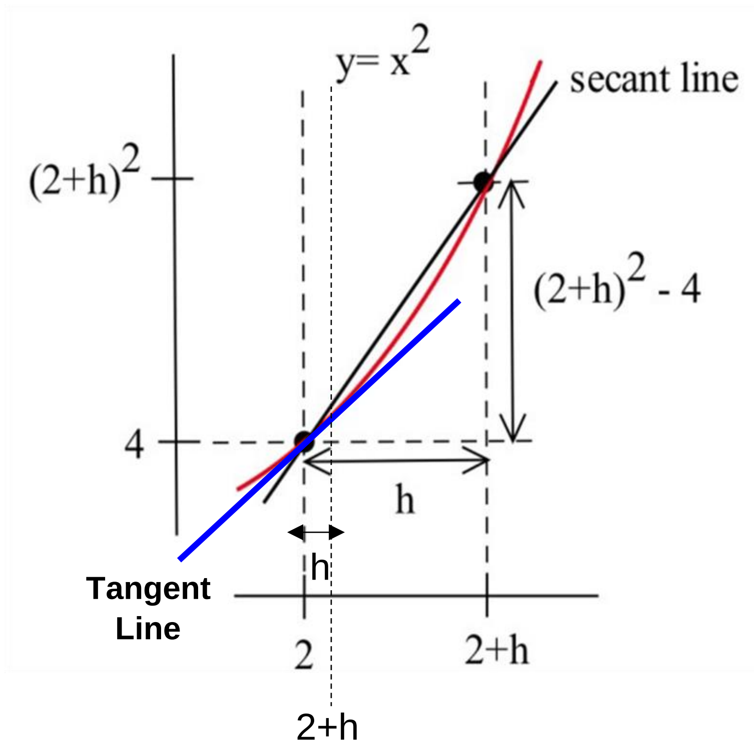
This is the same idea we developed in the last section when talking about limits. More formally, we could write: Slope of the tangent line:

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (4 + h) = 4$$

We can easily evaluate this limit using direct substitution, finding that as the interval (h) shrinks towards 0, the secant slope approaches the tangent slope, 4.

When we wanted to know how rapidly something is **changing at an instant in time**, it turns out we must find the **slope of a tangent line,** which we approximated with the **slope of a secant line**.

The slope of a tangent line gives us an **average rate of change** in the function between the given points.



The **Average Rate of Change** of a function on an interval, and the **Instantaneous rate of change** of a function at a point, is defined below.

An Average Rate of Change of a function on an interval $[x, x+h]$ can be found by finding the slope of the secant line between the points:

$$(x, f(x)) \quad \text{and} \quad (x+h, f(x+h))$$

Average rate of change (Secant Line)

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

An Average Rate of Change of a function on an interval $[a, b]$ can be found by finding the slope of the secant line, where $[b - a = h]$:

$$(x, (x+h)) = (a, b)$$

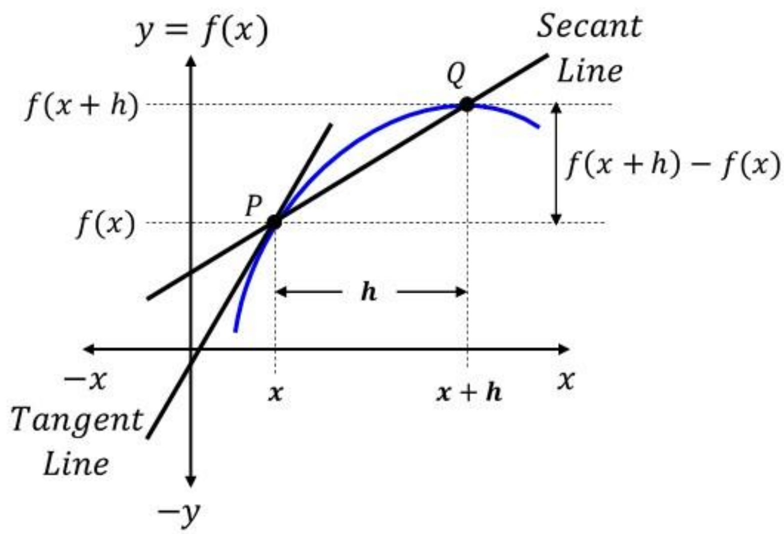
Average rate of change (Secant Line)

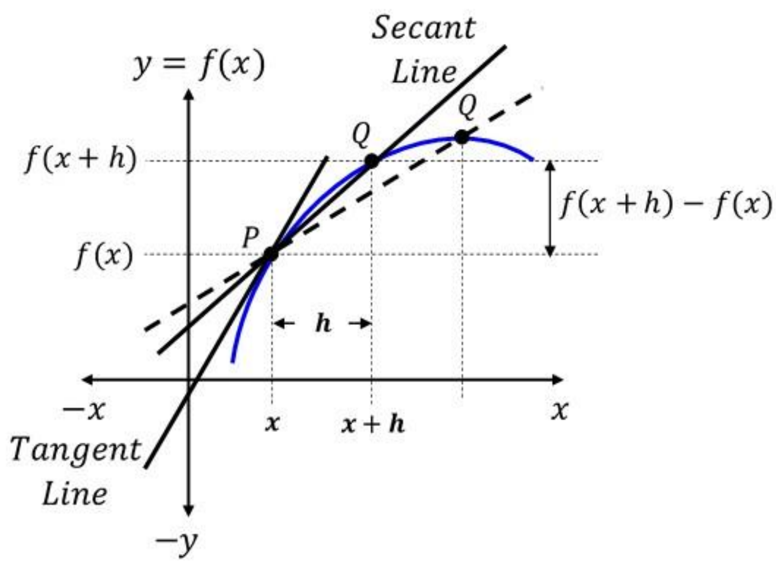
$$\frac{f(b) - f(a)}{b - a} = \frac{f(x+h) - f(x)}{h}$$

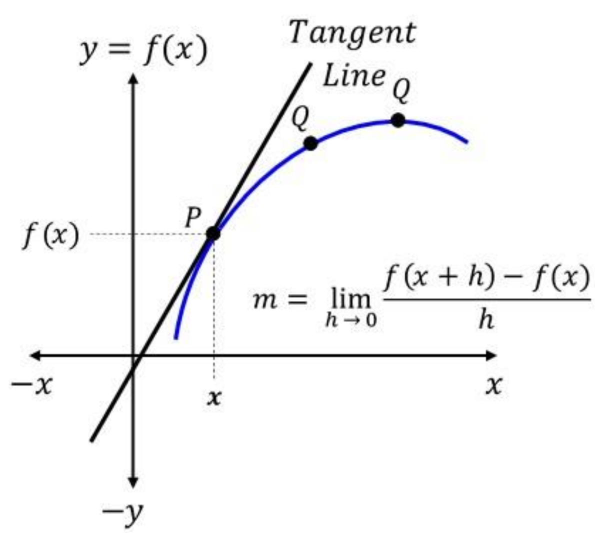
An Instantaneous rate of change of a function at a point $(x, f(x))$ can be found by finding the slope of the tangent line to the graph of (f) at $(x, f(x))$

Instantaneous rate of change (Tangent Line)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$


$y = f(x)$
Secant Line
Q
$f(x+h)$
$f(x+h) - f(x)$
P
$f(x)$
h
$-x$
x
$x+h$
x
Tangent Line
$-y$


Secant Line
$y = f(x)$
Q
Q
$f(x+h)$
$f(x+h) - f(x)$
P
$f(x)$
h
$-x$
x
$x+h$
x
Tangent Line
$-y$


Tangent Line
$y = f(x)$
Q
Q
P
$f(x)$
$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$-x$
x
x
$-y$

Example Problem #1

Given the function,

$$f(x) = x^2 + 1$$

Find

a) The **average rate of change** of ($f(x)$) on the interval [1, 3]

b) The **average rate of change** of ($f(x)$) on the interval $[1, 1+h]$

c) The **instantaneous rate of change** of ($f(x)$) when (x = 1)

Solution

a) The average rate of change on [1, 3] is the change in y-values with respect to the change in x-values, i.e. the slope of the secant line from $(1, f(1))$ to (3, f(3))

$$\frac{f(3) - f(1)}{3 - 1} = \frac{(3^2+1) - (1^2+1)}{3-1} = \frac{10-2}{2} = 4$$

b) On the interval $[1, 1+h]$, the ratio becomes:

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{((1+h)^2+1) - (1^2+1)}{h}$$

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{(1+2h+h^2+1) - 2}{h} = \frac{2h+h^2}{h} = 2+h$$

c) To find the instantaneous rate of change when x = 1, we need to find the slope of the tangent line to the graph at $(1, f(1))$, i.e, the limit of the above ratio as (h) approaches 0.

$$\lim_{h\to 0}\frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h\to 0}\frac{((1+h)^2+1) - (1^2+1)}{h}$$

$$\lim_{h\to 0}\frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h\to 0}(2+h) = 2$$

The Derivative

The Derivative:

The **derivative** of a function f at a point $(x,\ f(x))$ is the instantaneous rate of change in the function $(f(x))$ at that point $(x,\ f(x))$.

The **derivative** of a function f at a point $(x,\ f(x))$ is the slope of the tangent line to the graph of $(f(x))$ at that point $(x,\ f(x))$.

A function is called **differentiable** at $(x,\ f(x))$ if its derivative exists at $(x,\ f(x))$.

Derivative = Slope of tangent line = Instantaneous rate of change of a function at a point.

Notation for the Derivative:

The **derivative of ($\boldsymbol{y = f(x)}$) with respect to x** is written as:

$$f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

Which when read aloud is ("f prime of x"), or y' ("y prime"), equals $(\frac{dy}{dx})$, which is read aloud as ("dee why dee ex")

The notation that resembles a fraction is called **Leibniz notation**.

It displays not only the name of the function (f or y), but also the name of the variable (in this case, x).

It looks like a fraction because the derivative is a slope. In fact, this is simply,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

written in Roman letters instead of Greek letters.

The Derivative Cont'd:

When evaluating the derivative at a certain point, say ($x = a$), the notation becomes:

$$f'(a) = \frac{d}{dx} f(x)\Big|_{x=a}$$

Verb forms:

We **find the derivative** of a function, or **take the derivative** of a function, or **differentiate** a function.

Formal Algebraic Definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Practical Definition:

The derivative can be approximated by looking at an average rate of change, or the slope of a secant line, over a very tiny interval.

The tinier the interval, the closer this is to the true instantaneous rate of change or slope of the tangent line.

Looking Ahead: We will have methods for computing exact values of derivatives from formulas soon.

If the function is given to you as a table or graph, you will still need to approximate this way.

This is the foundation for the rest of this chapter. It's remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative (f')) will lead to so many important ideas and applications.

We now know how to find (or at least approximate) the derivative of a function for any x-value; this means we can think of the derivative as a function, too.

The inputs are the same x's; the output is the value of the derivative at that (x) value, i.e., the slope of the tangent line to the graph at that (x).

Example Problem #2

Given the function, find the derivative, using the definition of the derivative:

$$f(x) = 3x - 1$$

find the y-value of the function $(f(x))$ and the derivative $(f'(x))$ when (x = 4).

Solution:

y-value of the function

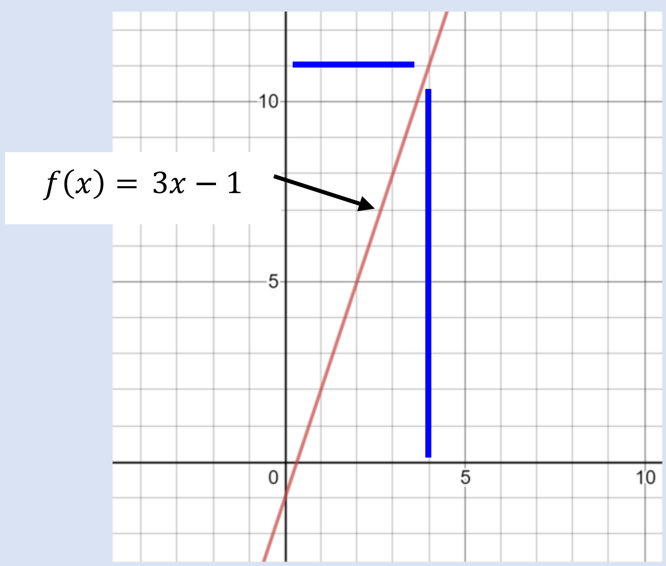
$$y = f(4) = 3(4) - 1 = 12 - 1 = 11$$

$$(x, y) = \big(4, f(4)\big) = (4, 11)$$

Derivative of the function

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{(3(4+h) - 1) - (3(4) - 1)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{12 + 3h - 1 - 12 + 1}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

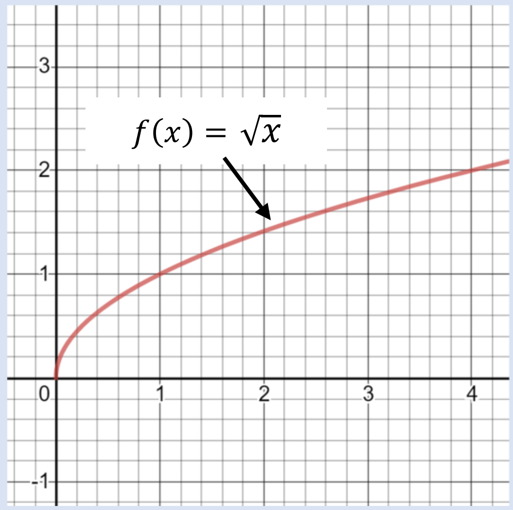


Example Problem #3

Given the function, find the derivative, using the definition of the derivative:

$$f(x) = \sqrt{x}$$

find the derivative of $(f(x))$ at any x-value, i.e. find the derivative function $(f'(x))$.



Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{(\sqrt{x+h)}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$f'(x) = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$f'(x) = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Please note that we used the method of rationalizing the numerator to simplify the ratio. Also, remember that we cannot substitute in (h) with (0), if the denominator still equals (*h*), i.e. we only evaluate the limit once we removed the (h) factor from the denominator!

Example Problem #3 – Cont'd

Given the derivative of the function $f'(x)$, and the function $f(x)$, find the values of the functions when: $(x = 1)$ and $(x = 4)$

Solution:

y-value of function:

$$y = f(x) = \sqrt{x}$$

Slope or Derivative of function:

$$m = f'(x) = \frac{1}{2\sqrt{x}}$$

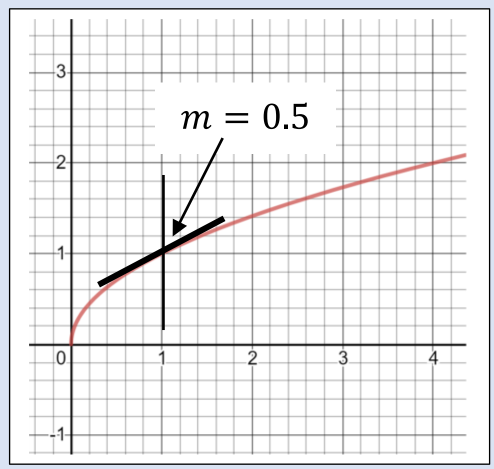
---

1. When $(x = -1)$ the values of the functions are:

$$y = f(1) = \sqrt{1} = 1$$

$$(x, y) = \big(1, f(1)\big) = (1, 1)$$

$$m = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} = 0.5$$



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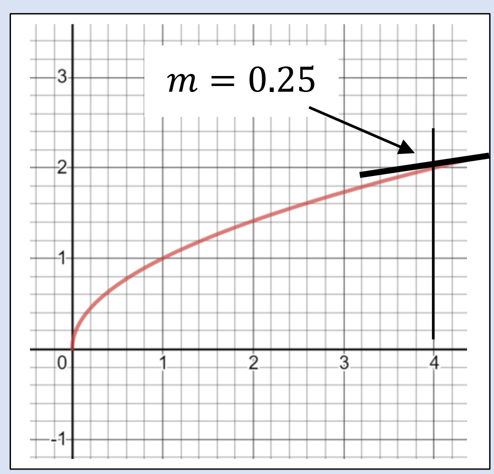
2. When $(x = 4)$ the values of the functions are:

$$y = f(4) = \sqrt{4} = 2$$

$$(x, y) = (4, f(4)) = (4, 2)$$

$$m = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4} = 0.25$$



Example Problem #4

Given the function, find the derivative, using the definition of the derivative:

$$f(x) = x^3 - 4$$

find the derivative of $(f(x))$ at any x-value, i.e. find the derivative function $(f'(x))$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^3 - 4) - (x^3 - 4)}{h}$$

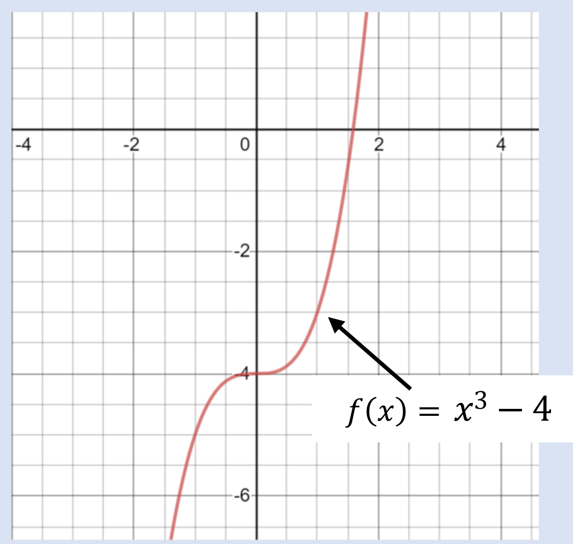
by distribution, we get:

$$f'(x) = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 4) - (x^3 - 4)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4 - x^3 + 4}{h}$$

canceling out common terms, we get:

$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$



factoring out h, we get:

$$f'(x) = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

reducing h, we get:

$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

Now, that h is no longer in the denominator, we can evaluate the limit by substituting h with 0.

$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

Example Problem #4 – Cont'd

Given the derivative of the function $f'(x)$, and the function $f(x)$, find the values of the functions when: ($x = -1$), ($x = 0$), ($x = 1$)

Solution:

y-value of function:

$$y = f(x) = x^3 - 4$$

Slope or Derivative of function:

$$m = f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

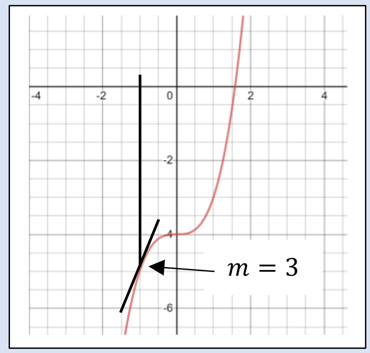
---

1. When ($x = -1$) the values of the functions are:

$$y = f(-1) = (-1)^3 - 4 = -1 - 4 = -5$$

$$(x, y) = (-1, f(-1)) = (-1, -5)$$

$$m = f'(-1) = 3(-1)^2 = (3)(1) = 3$$



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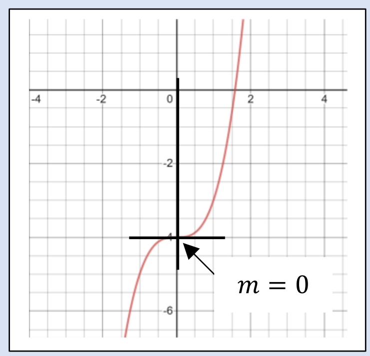
2. When ($x = 0$) the values of the functions are:

$$y = f(0) = (0)^3 - 4 = 0 - 4 = -4$$

$$(x, y) = (0, f(0)) = (0, -4)$$

$$m = f'(0) = 3(0)^2 = (3)(0) = 0$$



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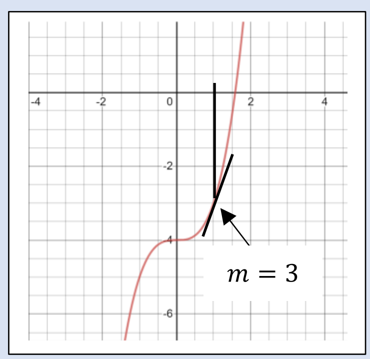
3. When ($x = 1$) the values of the functions are:

$$y = f(1) = (1)^3 - 4 = 1 - 4 = -3$$

$$(x, y) = (1, f(1)) = (1, -3)$$

$$m = f'(1) = 3(1)^2 = (3)(1) = 3$$



Example Problem #5

Suppose the temperature T in degrees Fahrenheit at a height x in feet above the ground is given by

$$y = T(x) = 0.002x^2 - 5x + 1 \quad \rightarrow \text{°F}$$

a) Find the derivative ($T'(x)$) of this function, and give an interpretation of it.

b) Find ($T'(100)$) and explain its meaning.

Solution

a) $T'(x)$

$$T'(x) = \lim_{h \to 0} \frac{T(x+h) - T(x)}{h}$$

$$T'(x) = \lim_{h \to 0} \frac{(0.002(x+h)^2 - 5(x+h) + 1) - (0.002x^2 - 5x + 1)}{h}$$

$$T'(x) = \lim_{h \to 0} \frac{0.002x^2 + 0.004xh + 0.002h^2 - 5x - 5h + 1 - 0.002x^2 + 5x - 1}{h}$$

$$T'(x) = \lim_{h \to 0} \frac{0.004xh + 0.002h^2 - 5h}{h}$$

$$T'(x) = \lim_{h \to 0}(0.004x + 0.002h - 5)$$

$$T'(x) = 0.004x - 5 \quad \rightarrow \frac{\text{°F}}{x}$$

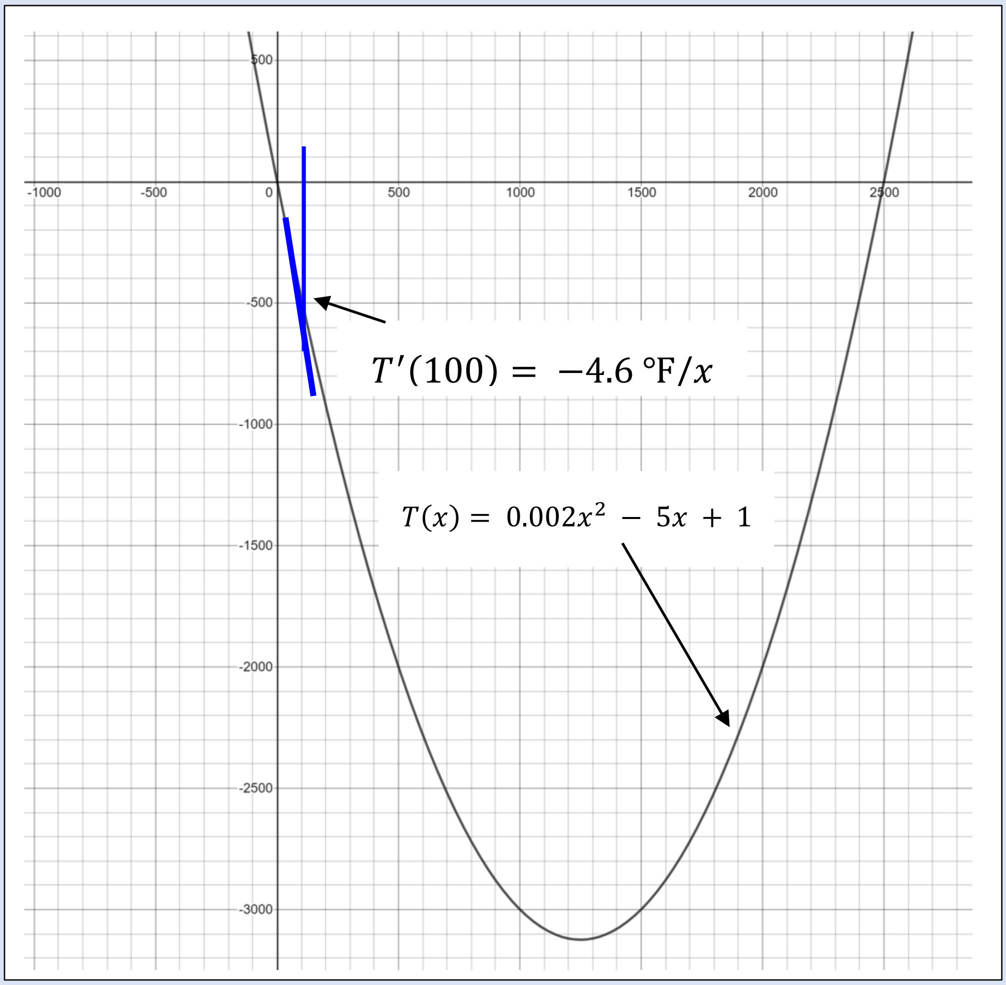
The change in temperature x feet above ground can be given by the expression ($0.004x - 5$) degrees Fahrenheit per additional foot.

Example Problem #5 – Cont'd

b) $T'(100)$

$$T'(100) = 0.004(100) - 5 = 0.4 - 5 = -4.6\,\frac{°\mathrm{F}}{x}$$

At an altitude of 100 feet above the ground, the temperature will decrease by 4.6 $\frac{°\mathrm{F}}{x}$ (Fahrenheit per each additional foot).

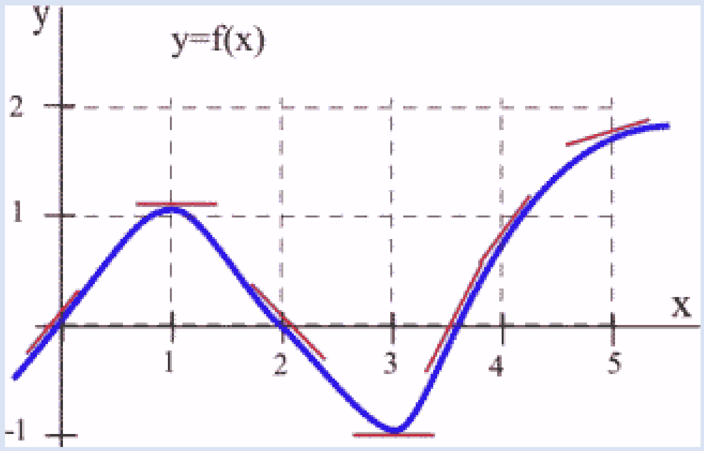


Example Problem #6

Given

Below is the graph of a function ($y = f(x)$).

We can use the information in the graph to fill in a table showing values of ($f'(x)$):



At various values of x, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points.

In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other.

Here are estimates for a few values of x (parts of the tangent lines used are shown above in the graph):

x	$y = f(x)$	$f'(x)$ = the estimated *slope* of the tangent line to the curve at the point (x, y).
0	0	1
1	1	0
2	0	-1
3	-1	0
3.5	0	1

Example Problem #6 – Cont'd

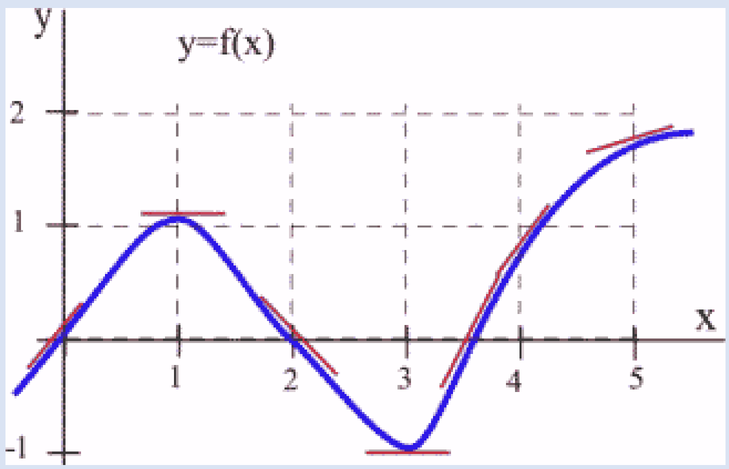
We can estimate the values of ($f'(x)$) at some non-integer values of (x), too:

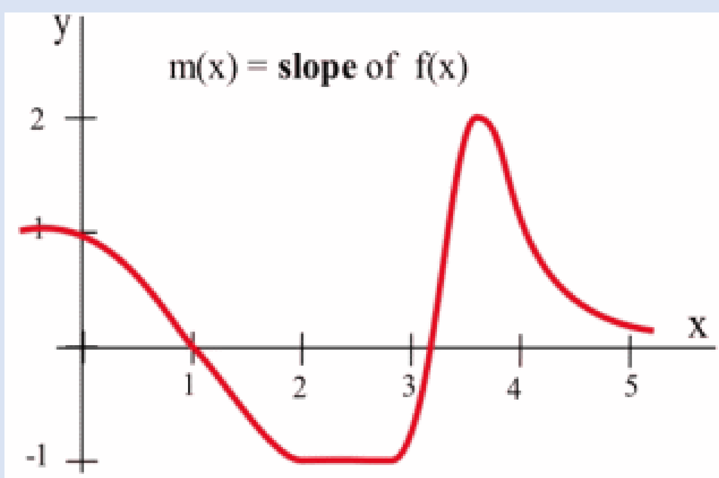
$$f'(0.5) \approx 0.5 \quad \text{and} \quad f'(1.3) \approx -0.3$$

We can even think about entire intervals.

For example, if ($0 < x < 1$), then ($f(x)$) is increasing, all the slopes are positive, and so ($f'(x)$) is positive.

The values of ($f'(x)$) definitely depend on the values of (x), and ($f'(x)$) is a function of (x). We can use the results in the table to help sketch the graph of ($f'(x)$).





1.2 - EXERCISES

1. Find the average rate of change of the function $f(x) = x^2$ from
a) x = 2 to x = 4
b) x = 2 to x = 3
c) x = 2 to x = 2.5
d) x = 2 to x = 2.1
e) x = 2 to x = 2.001

2. Find the average rate of change of the function $f(x) = x^2$ on the interval $[x, x+h]$

Use the **limit definition formula** introduced in this section for the following problems.

3.	Find the derivative of the function $f(x) = x^2 + 5$	4.	Find the derivative of the function $f(x) = x^3$
5.	Find the derivative of the function $f(x) = 3x^2 + 4x + 1$	6.	Find the derivative of the function $f(x) = 8$
7.	Find the derivative of the function $f(x) = \frac{1}{x}$	8.	Find the derivative of the function $f(x) = \sqrt{2x}$
9.	Find the derivative of the function $f(x) = 5x^2 - 3x$, at the point $x = 1$	10.	Find the derivative of the function $f(x) = 7x - 6$, at the point $x = 9$
11.	Find the instantaneous rate of change of $f(x) = x^2$ at $x = 2$	12.	Find the slope of the tangent line to the graph of $f(x) = \frac{2}{x-4}$, at the point $x = 5$

13.	The population of a certain city is given by the function $f(x) = x^3$, where x is years since 2000 and the population is measured in thousands of people. a) Find the average rate of change in the population between the years of 2000 and 2018. b) Find the instantaneous rate of change in population in the year 2018 and interpret this answer.

Solutions:

1. a) 6, b) 5, c) 4.5, d) 4.1, e) 4.001

2. $2x + h$

3. $f'(x) = 2x$

4. $f'(x) = 3x^2$

5. $f'(x) = 6x + 4$

6. $f'(x) = 0$

7. $f'(x) = -\frac{1}{x^2}$

8. $f'(x) = \frac{1}{\sqrt{2x}}$

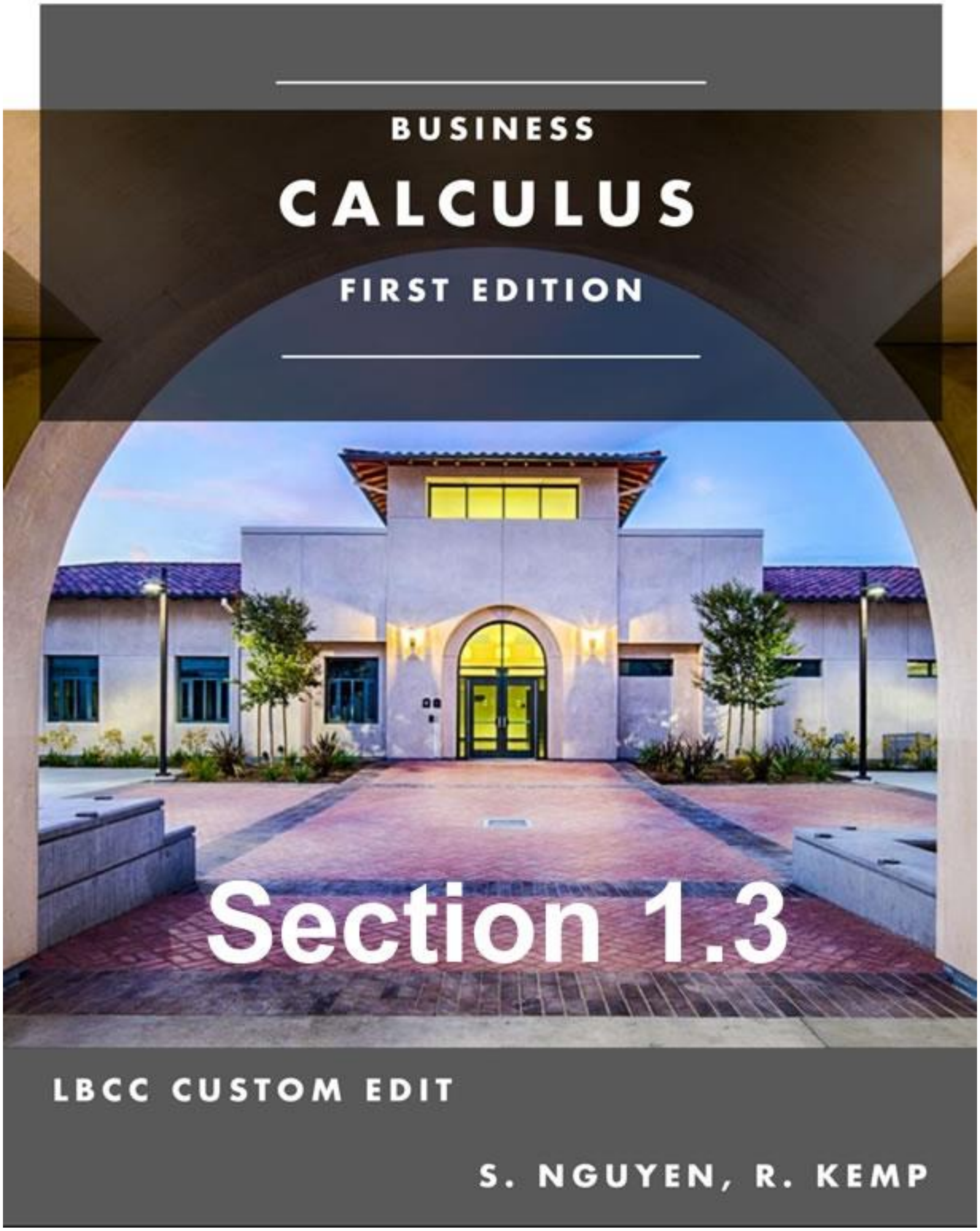
9. $f'(x) = 10x - 3$; $f'(1) = 7$

10. $f'(x) = 7$; $f'(9) = 7$

11. $f'(x) = 2x$; $f'(2) = 4$

12. $f'(x) = \frac{-2}{(x-4)^2}$; $f'(5) = -2$

13. a) 324 thousand people average increase per year between 2000 and 2018,
b) In year 2018, the population is increasing by 972 thousand people per year.


BUSINESS
CALCULUS
FIRST EDITION
Section 1.3
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S. NGUYEN, R. KEMP

1.3 - FORMULAS FOR DERIVATIVES

In this section, we'll get the derivative rules that will let us find formulas for derivatives when our function comes to us as a formula. Using the rules will be much easier than using the limit definition of the derivative that was introduced in the last section.

This is a very algebraic section, and you should get lots of practice.

Building Blocks

These are the simplest rules for the basic functions. We won't prove these rules; we'll just use them.

Derivative Rules: Building Blocks

In what follows, f and g are differentiable functions of x.

Derivative of a Constant (K) Rule:

$$\frac{d}{dx}K = 0$$

Constant Multiple Rule:

$$(Kf)' = \frac{d}{dx}Kf = K\frac{d}{dx}f = K(f)'$$

Sum or Difference Rule:

$$(f \pm g)' = \frac{d}{dx}(f \pm g) = \frac{d}{dx}f \pm \frac{d}{dx}g = f' \pm g'$$

Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Identity Function Rule:

$$\frac{d}{dx}(x) = 1$$

Here are all the basic rules in one place. The sum, difference, and constant multiple rule combined with the power rule allow us to easily find the derivative of any polynomial function.

But first, let's look at a few so that we can see they make sense.

Example Problem #1

Find the derivative of:

$$y = f(x) = 4$$

Solution:

Think about this one graphically too. The graph of *f* (x) = 4 is a horizontal line, so its slope equals zero.

$$y' = f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}4 = 0$$

Rule: The derivative of a constant is zero.

Example Problem #2

Find the derivative of:

$$y = f(x) = mx + b$$

Solution

This is a linear function, so its graph is its own tangent line! The slope of the tangent line, the derivative, is the slope of the line:

$$y' = f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(mx + b)$$

$$y' = f'(x) = m \cdot \frac{d}{dx}x + \frac{d}{dx}b = m \cdot 1 + 0$$

$$y = f'(x) = m$$

Rule: The derivative of a linear function is its slope.

Example Problem #3

Find the derivative of the function, and evaluate at ($f'(5)$):

$$y = f(x) = x^2$$

Solution:

a) We have solved this problem in the last section using our limit definition.

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x + 0$$

$$f'(x) = 2x$$

b) Now using the power rule

$$y = f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(x^2) = 2\cdot x^{2-1}$$

$$y = f'(x) = 2x$$

So, the derivative of (x^2) equals to ($2x$). Luckily, there is a handy rule we use to skip using the limit:

c) Now let's evaluate the derivative of the function at [$f'(5)$]

$$f'(5) = \frac{d}{dx}f(x)\bigg|_{x=5} = 2(5) = 10$$

Power Rule: The derivative *[$f(x) = x^n$]* is *[$f'(x) = nx^{n-1}$]*

Example Problem #4

Find the derivative of:

$$y = g(x) = 4x^3$$

Solution:

Using the power rule, we know that if ($f(x) = x^3$), then ($f'(x) = 3x^2$).

Notice that *($g(x)$)* is 4 times the function *($f(x)$)*. Think about what this change means to the graph of *($g(x)$)*; it's now 4 times as tall as the graph of ($f(x)$).

If we find the slope of a secant line, it will be,

$$\frac{\Delta g}{\Delta x} = \frac{4\Delta f}{\Delta x} = 4\frac{\Delta f}{\Delta x}$$

Each slope will be 4 times the slope of the secant line on the *f* graph.

This property will hold for the slopes of tangent lines, too:

$$y' = g'(x) = \frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3) = 4 \cdot [3x^2] = 12x^2$$

Rule: Constants come along for the ride; *(kf)'= k(f)'*

Example Problem #5

Find the derivative of:

$$f(x) = 5x^7 - 4x^5 + 0.5x^2 - 10x + 157$$

Solution:

$$f'(x) = \frac{d}{dx}(5x^7 - 4x^5 + 0.5x^2 - 10\text{x} + 157)$$

$$f'(x) = \frac{d}{dx}(5x^7) + \frac{d}{dx}(-4x^5) + \frac{d}{dx}(0.5x^2) + \frac{d}{dx}(-10x) + \frac{d}{dx}(157)$$

$$f'(x) = 5\frac{d}{dx}(x^7) - 4\frac{d}{dx}(x^5) + 0.5\frac{d}{dx}(x^2) - 10\frac{d}{dx}(x) + 0$$

$$f'(x) = 5(7x^6) - 4(5x^4) + 0.5(2x) - 10(1) + 0$$

$$f'(x) = 35x^6 - 20x^4 + x - 10$$

You don't have to show every single step. Do be careful when you're first working with the rules, but soon you'll be able to just write down the derivative directly.

Example Problem #6

Find the derivative of:

$$f(x) = 4x^3 + 2x^2 - 25x + 1 \qquad \textit{evaluate at} \quad f'(2)$$

Solution:

$$f'(x) = \frac{d}{dx}(4x^3 + 2x^2 - 25x + 1)$$

$$f'(x) = 4(3x^2) + 2(2x) - 25(1) + 0 = 12x^2 + 4x - 25$$

$$f'(2) = \frac{d}{dx}f(x)\bigg|_{x=2} = 12(2^2) + 4(2) - 25 = 31.$$

The power rule works even if the power is negative or a fraction.

To apply it, first translate all roots and basic rational expressions into exponents.

Recall from algebra that:

Review of some algebra rules needed in the following examples:

$$\frac{1}{x^n} = x^{-n}$$

$$\sqrt[n]{x^m} = x^{m/n}$$

Example Problem #7

Find the derivative of the function:

$$f(x) = \sqrt[3]{x} + \frac{6}{\sqrt[5]{x^7}}$$

Solution:

First, we will rewrite the function using powers:

$$f(x) = x^{\frac{1}{3}} + 6x^{-\frac{7}{5}}$$

Then the derivative can be found using the power rule:

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{3}} + 6x^{-\frac{7}{5}}\right) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) + 6\frac{d}{dx}(x^{-\frac{7}{5}})$$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} + 6\cdot\left(-\frac{7}{5}x^{-\frac{7}{5}-1}\right) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{42}{5}x^{-\frac{12}{5}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{42}{5}x^{-\frac{12}{5}} = \frac{1}{3\sqrt[3]{x^2}} - \frac{42}{5\sqrt[5]{x^{12}}}$$

Changes in Cost, Revenue and Profit

Derivatives are useful in analyzing changes in cost, revenue, and profit.

The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives.

The marginal cost is the derivative of the cost function.

The marginal revenue is the derivative of the revenue function.

The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

If ($C(x)$) is the **cost** of producing x items, then the marginal cost ($MC(x)$) is:

$$MC(x) = C'(x) = \frac{d}{dx}C(x)$$

If ($R(x)$) is the **revenue** obtained from selling x items, then the marginal revenue ($MR(x)$) is:

$$MR(x) = R'(x) = \frac{d}{dx}R(x)$$

If ($P(x) = R(x) - C(x)$) is the profit obtained from selling (x) items, then the marginal profit MP(x) is defined to be:

$$MP(x) = MR(x) - MC(x) = R'(x) - C'(x)$$

$$MP(x) = P'(x) = \frac{d}{dx}R(x) - \frac{d}{dx}C(x)$$

($MC(x) = C'(x)$) approximates the cost of producing one additional item, ($MR(x) = R'(x)$) approximates the revenue obtained by selling one additional item, and ($MP(x) = P'(x)$) approximates the profit obtained by producing and selling one additional item.

Example Problem #8

The cost of producing x items if ($C(x)$) hundred dollars.

$$C(x) = \sqrt{x} \quad \textit{hundred dollars}$$

a) What is the cost for producing 100 items? 101 items? What is the cost of the 101st item?

b) For C(x) = $\sqrt{x}$, calculate the marginal cost MC(x) = C' (x) and evaluate it at (x = 100). How does your answer compare with the last answer in part a)?

Solution:

a) C(100) = $\sqrt{100} = 10$ hundred dollars, so $1,000.

C(101) = $\sqrt{101} = 10.049875$ hundred dollars, so $1,004.99,

This means the cost of producing the 101st item is [$1004.99 - $1000 = $4.99].

Note that this is exactly the average change in cost from the 100th to the 101st item

$$\frac{C(101) - C(100)}{101 - 100} = \frac{1004.99 - 1000}{1} = 4.99$$

b) Using the power rule, we find ($C'(x) \ = \ MC(x)$)

$$C'(x) \ = \ MC(x) \ = \ \frac{d}{dx}(x^{\frac{1}{2}}) \ = \ \frac{1}{2}x^{-\frac{1}{2}} \ = \ \frac{1}{2\sqrt{x}}$$

$C'(100) \ = \ \frac{1}{2\sqrt{100}} = 0.05$ hundred dollars, so $5.

Look how close the answers are!

This proves again that the derivative, which is the instantaneous rate of change at a point, approximates an average change at that point, if we keep the x-values close together. In terms of the derivative of the cost function, as stated above, the marginal cost approximates the cost of producing one more item.

Example Problem #9

Find the equation of the tangent line to the graph of:

$$f(x) = 10 - x^2 \qquad when \quad x = 2$$

Solution:

The (y_1) **point** at which the tangent line touches the graph at the value (x=2).

$$y_1 = f(2) = 10 - (2)^2 = 10 - 4 = 6$$

To find the equation of any line we need to have the **slope** m of the line and **one point** on that line (x_1, y_1).

Then we can use the point-slope formula to find the equation of the line:

$$(y - y_1) = m(x - x_1)$$

The **slope** of the tangent line to the graph of any function equals to the derivative of that function. We need specifically the equation of the tangent line when x = 2, so we need to find the derivative of $(f(x))$ and evaluate it at x = 2.

$$f'(x) = -2x$$

$$m = f'(2) = -2(2) = -4$$

The **point** at which the tangent line touches the graph has the x-value 2.

This is a common point for both the graph and the tangent line, so the point we will be using is:

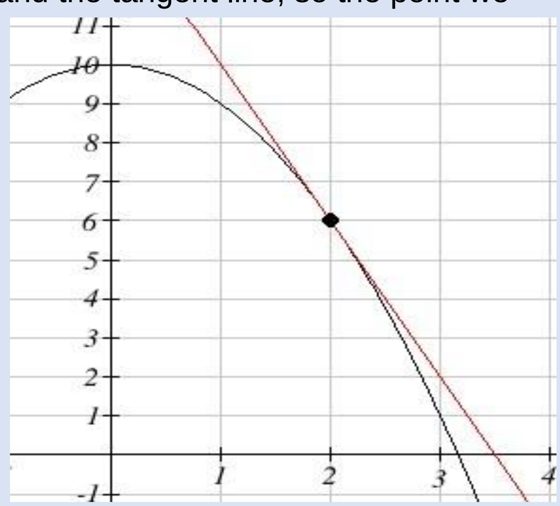
$$(x_1, y_1) = (x_1, f(2)) = (2, 6)$$

Now using the point-slope formula, we get

$$(y - 6) = -4(x - 2)$$

$$y - 6 = -4x + 8$$

$$y = -4x + 14$$



So, the equation of the tangent line is $(y = -4x + 14)$. Graphing, we can verify this line is indeed the tangent line to the graph.

1.3 - EXERCISES

Solve the following problems using the formulas we learned in this section.			
Find the derivative of the following functions:			
1.	$f(x) = x^5$	2.	$f(x) = 6x^4$
3.	$f(x) = 5x + 7$	4.	$f(x) = 4x^3 - 7x^2 + 6x - 9$
5.	$f(x) = \sqrt[3]{x}$	6.	$f(x) = \dfrac{1}{x^4}$
7.	$y = \sqrt[4]{x^5} - \dfrac{1}{\sqrt{x}}$	8.	$y = 3x^8 + \dfrac{22}{x^{10}}$
9.	$f(x) = (2x - 3)^2$	10.	$y = \dfrac{2x^3 - 12x}{2x}$
11.	$y = 7\sqrt{x} - 9\sqrt[3]{x} + 8\sqrt[4]{x^7} + 9x - 10$		
12.	$f(x) = \dfrac{20x^6 - 14x^5 + 2x^4 - 9x^3 - 2x + 10}{4x^3}$		

13.	Find the equation of the tangent line to the graph of $f\ (x) = 9x^3 - 5x$ at the point $x = 1$.
14.	**Using the answer from problem #3**, find the equation of the tangent line to the graph of $f\ (x) = 5x + 7$ at the point $x = 3$.
15.	**Using the answer from problem #9**, find the equation of the tangent line to the graph of $f(x) = (2x - 3)^2$ at the point $x = 5$.
16.	Given the cost function of producing x items $C(x) = 5x + 100$, the revenue function of selling x items is $R(x) = 3x^2$, both in hundreds of dollars, find: a) The profit function b) The marginal cost, marginal revenue and marginal profit functions. c) Evaluate the functions you found in part b) at x = 100 and interpret the answers.
17.	A stuntman estimates the time T, in seconds, for him to fall x meters, is $T(x) = 0.54\sqrt{x}$. a) Find the time of fall in seconds when $x = 9$ meters. b) Find the instantaneous rate of change of $T(x)$ with respect to x when $x = 9$ meters. c) Find the equation of a tangent line of $T(x)$ when $x = 9$ meters.
18.	A book publisher has a cost function given by $C(x) = \frac{x^3 + 2x + 3}{x^2}$, where x is the number of copies of a book and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

19.	The daily profit function, in dollars, for selling x cars, at a dealership is $P(x) = -0.006x^3 - 0.3\,x^2 + 700x - 1000$. Currently about 50 cars are being sold daily. a) Find the current profit of selling 50 cars daily. b) Find the marginal profit for selling 50 cars daily and interpret the answer. c) Find the profit obtained from selling 51 cars daily and compare the answer to part b). How are the answers related?

Solutions:

1. $f'(x) = 5x^4$

2. $f'(x) = 24x^3$

3. $f'(x) = 5$

4. $f'(x) = 12x^2 - 14x + 6$

5. $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

6. $f'(x) = \frac{-4}{x^5}$

7. $\frac{dy}{dx} = \frac{5}{4}\sqrt[4]{x} + \frac{1}{2\sqrt{x^3}}$

8. $\frac{dy}{dx} = 24x^7 - \frac{220}{x^{11}}$

9. $f'(x) = 8x - 12$

10. $\frac{dy}{dx} = 2x$

11. $\frac{dy}{dx} = \frac{7}{2\sqrt{x}} - \frac{3}{\sqrt[3]{x^2}} + 14\sqrt[4]{x^3} + 9$

12. $f'(x) = 15x^2 - 7x + \frac{1}{2} + \frac{1}{x^3} - \frac{15}{2x^4}$

13. $y = 22x - 18$

14. $y = 5x + 7$

15. $y = 28x - 91$

16. a) $P(x) = 3x^2 - 5x - 100$
 b) $MC(x) = 5, MR(x) = 6x, MP(x) = 6x - 5$
 c) $MC(100) = 5, MR(100) = 600, MP(100) = 595$

After producing 100 items, the cost per item increases by \$500/item, the revenue increases by \$60000/item and the profit increases by \$59500/item.

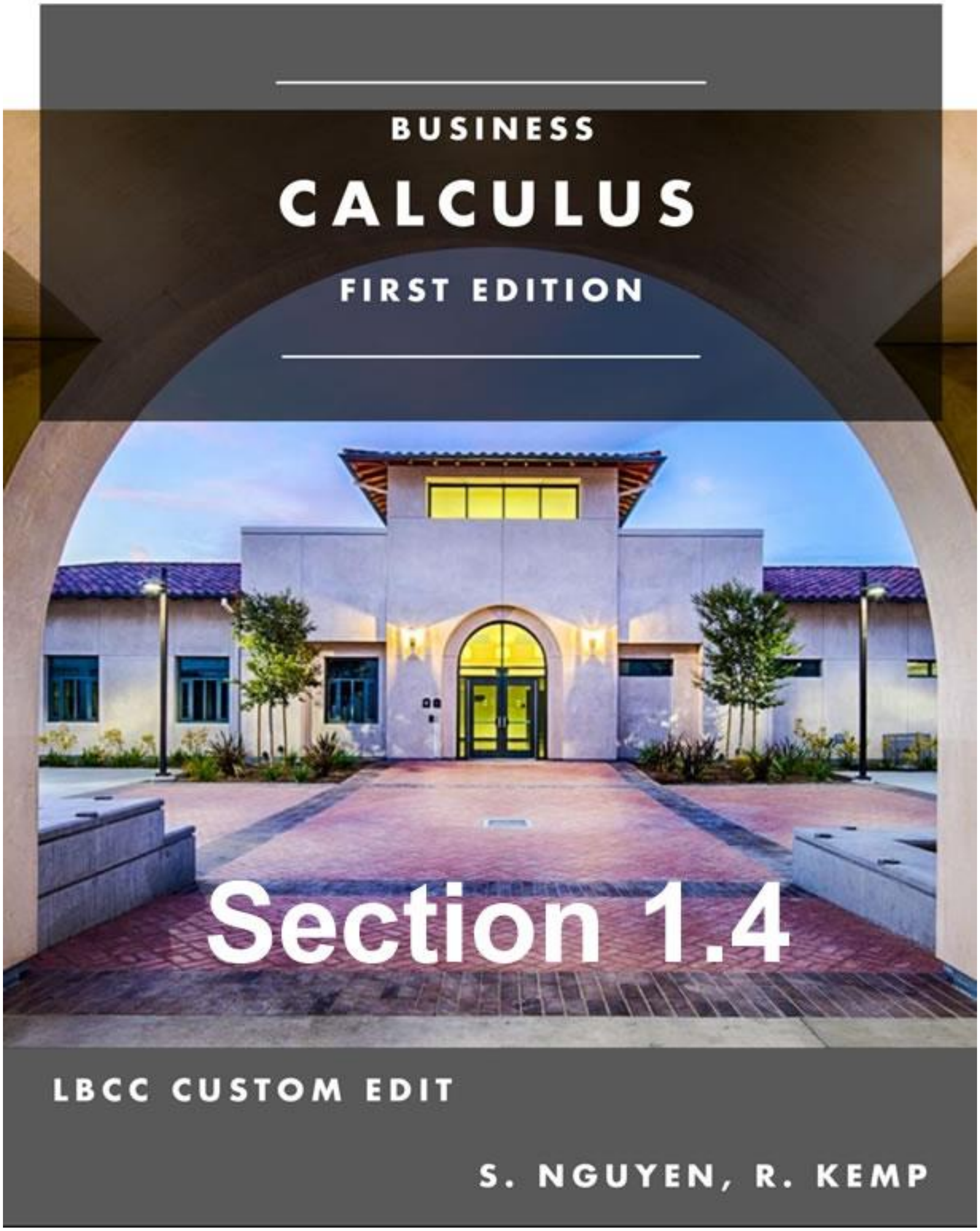
17. a) $T(9) = 1.62\ sec$
b) $T'(x) = \frac{0.54}{2\sqrt{x}}$, $T'(9) = 0.09\ sec/m$,
After falling for 9 meters, the stuntman's fall takes 0.09 extra seconds for every meter that he falls.
c) $T(x) = 0.09x + 0.81$

18. $C'(x) = 1 - \frac{2}{x^2} - \frac{6}{x^3}, C'(2) = -0.25$.

After 2 books have been produced the cost per book decreases at a rate of 25 cents/book.

19. a) $P(50) = 32{,}500\ dollars$.
b) $MP(x) = -0.018x^2 - 0.6x + 700, MP(50) = 625\ dollars/car$
c) $P(51) = 33{,}123.79\ dollars$.

We see that when selling one additional car after 50, the profit increases by \$623.79, which is very close to the number we got from the marginal profit after 50 cars. So, the instantaneous rate of change approximates the average rate of change.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1.4 - PRODUCT AND QUOTIENT RULES

The basic rules will let us tackle simple functions.

But what happens if the functions get more complicated? In this section, we will tackle functions that are made up of the product or quotient of other functions.

Example Problem #1

Find the derivative of the function:

$$f(x) = (4x^3 - 11)(x + 3)$$

Solution:

One way to solve this problem, would be to first multiply out the parentheses, as we did in the last section.

$$f(x) = 4x^4 + 12x^3 - 11x - 33$$

Now since we are dealing with a sum of terms, we can use the Sum/Difference Rule:

$$f'(x) = \frac{d}{dx}4x^4 + \frac{d}{dx}12x^3 - \frac{d}{dx}11x - \frac{d}{dx}33$$

$$f'(x) = 4\frac{d}{dx}x^4 + 12\frac{d}{dx}x^3 - 11\frac{d}{dx}x - \frac{d}{dx}33$$

$$f'(x) = 4(4x^3) + 12(3x^2) - 11 - 0$$

$$f'(x) = 16x^3 + 36x^2 - 11$$

Now suppose we wanted to find the derivative of

$$f(x) = (4x^5 + x^3 - 1.5x^2 - 11)(x^7 - 7.25x^5 + 120x + 3)$$

This function is not a simple sum or difference of polynomials. It's a product of polynomials. We **could** simply multiply it out to find its derivative as before – who wants to volunteer? Nobody?

We'll need a rule for finding the derivative of a product so we don't have to multiply everything out.

It would be great if we can just take the derivatives of the factors and multiply them, but unfortunately that won't give the right answer.

To see that, consider finding derivative of

$$f(x) = (4x^3 - 11)(x + 3)$$

We already worked out the derivative. It's ($f'(x) = 16x^3 + 36x^2 - 11$).

What if we try differentiating the factors and multiplying them?

We'd get ($(12x^2)(1) = 12x^2$) which is totally different from the correct answer.

The rules for finding derivatives of products and quotients are a little complicated, but they save us the much more complicated algebra we might face if we were to try to multiply things out.

They also let us deal with products where the factors are not polynomials and therefore might not be able to be multiplied out first.

We can use these rules, together with the basic rules, to find derivatives of many complicated looking functions.

Derivative Rules: Product and Quotient Rules

In what follows, $\boldsymbol{f}$ and $\boldsymbol{g}$ are differentiable functions of *x*.

Product Rule: $(\boldsymbol{fg})' = \boldsymbol{f'g} + \boldsymbol{fg'}$

Leibniz Form - Product Rule:

$$\frac{d}{dx}(\boldsymbol{f}\cdot\boldsymbol{g}) = \boldsymbol{g}\cdot\left(\frac{d}{dx}\boldsymbol{f}\right) + \boldsymbol{f}\cdot\left(\frac{d}{dx}\boldsymbol{g}\right)$$

The derivative of the first factor times the second factor left alone, plus the first factor left alone times the derivative of the second factor.

The product rule can extend to a product of several functions; the pattern continues – take the derivative of each factor in turn, multiplied by all the other factors left alone, and add them up.

Quotient Rule:

$$\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)' = \frac{\boldsymbol{f'g} - \boldsymbol{fg'}}{\boldsymbol{g}^2}$$

Leibniz Form - Quotient Rule:

$$\frac{d}{dx}\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right) = \frac{\boldsymbol{g}\cdot\left(\frac{d}{dx}\boldsymbol{f}\right) - \boldsymbol{f}\cdot\left(\frac{d}{dx}\boldsymbol{g}\right)}{\boldsymbol{g}^2}$$

The numerator of the result resembles the product rule, but there is a minus instead of a plus; the minus sign goes with the g'.

The denominator is simply the square of the original denominator – no derivatives there.

Example Problem #2

Find the derivative of the function using the Product rule:

$$f(x) = 4x^3x^8$$

Solution:

a) Using the Product Rule and the Power rule:

$$(fg)' = f'g + f\,g'$$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(4x^3x^8) = 4\cdot\frac{d}{dx}(x^3x^8)$$

$$f'(x) = 4\cdot\left[x^3\frac{d}{dx}(x^8) + x^8\frac{d}{dx}(x^3)\right]$$

$$f'(x) = 4\cdot\left[x^3[8x^{8-1}] + x^8\frac{d}{dx}[3x^{3-1}]\right]$$

$$f'(x) = 4\cdot\left[x^3[8x^7] + x^8[3x^2]\right]$$

$$f'(x) = 4\cdot[8x^3x^7 + 3x^8x^2]$$

$$f'(x) = 4\cdot[8x^{10} + 3x^{10}]$$

$$f'(x) = 4\cdot[11x^{10}] = 44x^{10}$$

b) Using Algebra and the Power rule:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(4x^3x^8) = 4\cdot\frac{d}{dx}(x^3x^8)$$

$$f'(x) = 4\cdot\frac{d}{dx}(x^{11}) = 4\cdot[11x^{11-1}] = 44x^{10}$$

Example Problem #3

Find the derivative of the function using the Product rule:

$$f(x) = (4x^3 - 11)(x + 3)$$

Solution:

Using the Product Rule:

$$(fg)' = f'g + f g'$$

$$f'(x) = (x + 3)\frac{d}{dx}(4x^3 - 11) + (4x^3 - 11)\frac{d}{dx}(x + 3)$$

$$f'(x) = (x + 3)(12x^2 - 0) + (4x^3 - 11)(1 + 0)$$

$$f'(x) = 12x^2(x + 3) + (4x^3 - 11)$$

$$f'(x) = 12x^3 + 36x^2 + 4x^3 - 11$$

$$f'(x) = 16x^3 + 36x^2 - 11$$

Example Problem #4

Find the derivative of the function using the Product Rule:

$$f(x) = (4x^5 + x^3 - 1.5x^2 - 11)(x^7 - 7.25x^5 + 120x + 3)$$

Solution:

Using the Product Rule:

$$(fg)' = f'g + f\,g'$$

$$f'(x) = (x^7 - 7.25x^5 + 120x + 3)\frac{d}{dx}(4x^5 + x^3 - 1.5x^2 - 11) + (4x^5 + x^3 - 1.5x^2 - 11)\frac{d}{dx}(x^7 - 7.25x^5 + 120x + 3)$$

$$f'(x) = (x^7 - 7.25x^5 + 120x + 3)(20x^4 + 3x^2 - 3x) + (4x^5 + x^3 - 1.5x^2 - 11)(7x^6 - 36.25x^4 + 120)$$

Now, this answer would be even harder to multiply than the initial function. Leave the answer in the product form, unless the multiplication is easy to perform or there is a reason to simplify further. We found the derivative.

Example Problem #5

Find the derivative of:

$$y = \frac{x^3 + 1}{x + 2}$$

Solution:

We cannot reduce this fraction, as we did in some of the examples in the previous section, because we are dividing by more than one term.

We are going to use the Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

$$\frac{dy}{dx} = \frac{(x+2)\cdot\frac{d}{dx}(x^3+1) - (x^3+1)\cdot\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{(x+2)(3x^2) - (x^3+1)(1)}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{3x^3 + 6x^2 - x^3 - 1}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{2x^3 + 6x^2 - 1}{(x+2)^2}$$

Example Problem #6

Find the derivative of:

$$y = \frac{(x-3)(2x^3+5x)}{(3x^2-x)}$$

Solution:

Looking at this function, we see that it has both a multiplication and a division, so we would need to use both the Product and the Quotient Rules.

This would turn pretty nasty, quickly.

We can at least try to avoid using the Product Rule by multiplying the polynomials in the numerator. This is easy to do.

$$y = \frac{2x^4 - 6x^3 + 5x^2 - 15x}{(3x^2-x)}$$

The division is not easy to perform, so we should use the Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

$$\frac{dy}{dx} = \frac{(8x^3-18x^2+10x-15)(3x^2-x)-(2x^4-6x^3+5x^2-15x)(6x-1)}{(3x^2-x)^2}$$

It would take a while to simplify this answer further, so we will leave it as it is.

If the algebra allows to easily perform the multiplication or division and you can avoid using the Product or Quotient Rules, then do that first.

If is much easier not to have to use these rules.

Average Cost, Revenue and Profit

Let's take again a look at the topics of cost, revenue and profit that we have discussed in the last section.

Many times, the cost of producing certain items is made up of a cost per item (or variable cost) plus fixed costs.

The **Total Variable Cost (TVC)** for (x) items is the amount of money you spend to actually produce them.

TVC includes things like the materials you use, the electricity to run the machinery, gasoline for your delivery vans, maybe the wages of your production workers. These costs will vary according to, how many items you produce.

$$C(x) = f(x)$$

The **Fixed Cost (FC)** is the amount of money you have to, spend regardless of how many items you produce.

FC can include things like rent, purchase costs of machinery, and salaries for office staff. You have to pay the fixed costs even if you don't produce anything.

$$C(0) = K$$

The **Total Cost (TC**, or sometimes just **C)** for (x) items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing x items.

$$C(x) = f(x) + C(0)$$

The **Average Cost (AC)** for (x) items is the total cost $(C(x))$ divided by (x). You can also talk about the average fixed cost, FC/x, or the average variable cost, TVC/x.

$$AC(x) = \frac{C(x)}{x} = \frac{f(x) + C(0)}{x} = \frac{f(x)}{x} + \frac{C(0)}{x}$$

The **Marginal Average Cost (AC)** for (x) items is the derivative of the **Average Cost (AC)** for (x) items.

$$MAC(x) = \frac{d}{dx}[AC(x)] = \frac{d}{dx}\left[\frac{C(x)}{x}\right]$$

Your **Revenue** is the amount of money you actually take in from selling your products. **Revenue = price × quantity.**

The **Total Revenue (TR**, or just **R)** for x items is the total amount of money you take in for selling x items.

$$R(x) = p(x) \cdot x$$

The **Average Revenue (AR)** for x items is the total revenue divided by x, or **R**/x.

$$AR(x) = \frac{R(x)}{x}$$

The **Marginal Average Revenue (MAR)** for (x) items is derivative of the **Average Revenue (AR)** for (x) items.

$$MAR(x) = \frac{d}{dx}[AR(x)] = \frac{d}{dx}\left[\frac{R(x)}{x}\right]$$

The **Profit (P)** for x items is R(x) – C(x), the difference between total revenue and total costs.

$$P(x) = R(x) - C(x)$$

The **Average Profit** for x items is **P**/x.

$$AP(x) = \frac{P(x)}{x} = \frac{R(x) - C(x)}{x}$$

The **Marginal Average Profit (MAP)** for (x) items is derivative of the **Average Profit (AP)** for (x) items.

$$MAP(x) = \frac{d}{dx}[AP(x)] = \frac{d}{dx}\left[\frac{P(x)}{x}\right] = \frac{d}{dx}\left[\frac{R(x) - C(x)}{x}\right]$$

Example Problem #7

The cost of producing (x) bicycles is ($C(x) = 0.2x + 5$) thousands of dollars.

a) Find the average cost per bicycle.

b) Evaluate the average cost when x = 50 and interpret the answer.

c) Find the marginal average cost.

d) Evaluate the marginal average cost when x = 50 and interpret the answer.

$$C(x) = 0.2x + 5$$

Solution:

a) Average Cost per bicycle:

$$AC(x) = \frac{0.2x + 5}{x} = 0.2 + \frac{5}{x}$$

b) Average Cost when (x = 50)

$$AC(50) = \frac{0.2(50) + 5}{50} = 0.2 + \frac{5}{50} = 0.3$$

When 50 bicycles are produced, the average cost of producing one bicycle is 0.3 thousand of dollars, i.e. $300/bicycle.

c) Marginal Average Cost per bicycle:

$$MAC(x) = \frac{d}{dx}\left(\frac{0.2x + 5}{x}\right) = \frac{(0.2)(x) - (0.2x + 5)(1)}{x^2} = \frac{-5}{x^2}$$

d) Marginal Average Cost when (x = 50)

$$MAC(50) = \frac{-5}{(50)^2} = -0.002$$

When 50 bicycles are produced the average cost of producing one additional bicycle decreases by 0.002 thousand of dollars per additional bicycle, i.e. $2/bicycle.

Note that the total costs of producing the bicycles are rising, but it is getting cheaper and cheaper to produce each bicycle as we produce more.

The average cost of producing the 51 bicycle is expected to only be $298/ bicycle.

Example Problem #8

Differentiate the function using the appropriate Derivative Rule.

$$h(x) = x^2 \cdot x^6$$

Solution:

a) Use the Product Rule:

$$(fg)' = f'g + f\,g'$$

$$h'(x) = \frac{d}{dx}(x^2 \cdot x^6)$$

$$h'(x) = x^2 \cdot \frac{d}{dx}(x^6) + x^6 \cdot \frac{d}{dx}(x^2)$$

$$h'(x) = x^2 \cdot [6 \cdot x^{6-1}] + x^6 \cdot [2 \cdot x^{2-1}]$$

$$h'(x) = x^2 \cdot [6 \cdot x^5] + x^6 \cdot [2 \cdot x]$$

$$h'(x) = 6 \cdot x^{5+2} + 2 \cdot x^{6+1}$$

$$h'(x) = 6 \cdot x^7 + 2 \cdot x^7$$

$$h'(x) = 8 \cdot x^7$$

b) Use Algebra:

$$h'(x) = \frac{d}{dx}(x^2 \cdot x^6) = \frac{d}{dx}(x^8) = 8 \cdot x^{8-1} = 8 \cdot x^7$$

Example Problem #9

Differentiate the function using the appropriate Derivative Rule.

$$h(x) = \frac{x^6}{x^2}$$

Solution:

a) Use the Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

$$h'(x) = \frac{d}{dx}\left(\frac{x^6}{x^2}\right)$$

$$h'(x) = \frac{x^2 \cdot \frac{d}{dx}(x^6) - x^6 \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$h'(x) = \frac{x^2 \cdot [6 \cdot x^{6-1}] - x^6 \cdot [2 \cdot x^{2-1}]}{x^4}$$

$$h'(x) = \frac{x^2 \cdot [6 \cdot x^5] - x^6 \cdot [2 \cdot x]}{x^4}$$

$$h'(x) = \frac{6 \cdot x^{5+2} - 2 \cdot x^{6+1}}{x^4} = \frac{6 \cdot x^7 - 2 \cdot x^7}{x^4} = \frac{4 \cdot x^7}{x^4}$$

$$h'(x) = 4 \cdot \left(\frac{x^7}{x^4}\right) = 4x^3$$

b) Use Algebra:

$$h'(x) = \frac{d}{dx}\left(\frac{x^6}{x^2}\right) = \frac{d}{dx}(x^4) = 4 \cdot x^{4-1} = 4x^3$$

Example Problem #10

Differentiate the function using the appropriate Derivative Rule.

$$f(x) = \frac{x^3}{x^4 + 5}$$

Solution:

a) Use the Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^3}{x^4 + 5}\right)$$

$$f'(x) = \frac{(x^4 + 5)\cdot\frac{d}{dx}(x^3) - x^3\cdot\frac{d}{dx}(x^4 + 5)}{(x^4 + 5)^2}$$

$$f'(x) = \frac{(x^4 + 5)\cdot[3\cdot x^{3-1}] - x^3\cdot[4\cdot x^{4-1} + 0]}{(x^4 + 5)^2}$$

$$f'(x) = \frac{(x^4 + 5)\cdot[3\cdot x^2] - x^3\cdot[4\cdot x^3]}{(x^4 + 5)^2}$$

$$f'(x) = \frac{x^4\cdot[3\cdot x^2] + 5\cdot[3\cdot x^2] - x^3\cdot[4\cdot x^3]}{(x^4 + 5)^2}$$

$$f'(x) = \frac{3x^6 + 15\cdot x^2 - 4x^6}{(x^4 + 5)^2}$$

$$f'(x) = \frac{-x^6 + 15\cdot x^2}{(x^4 + 5)^2} = \frac{-x^2\cdot(x^4 - 15)}{(x^4 + 5)^2}$$

Example Problem #11

Differentiate the function using the appropriate Derivative Rule.

$$f(t) = (t^2 - 3) \cdot \sqrt{t}$$

Solution:

a) Use the Product Rule:

$$(fg)' = f'g + f\,g'$$

$$f'(t) = \frac{d}{dt}\left((t^2 - 3) \cdot \sqrt{t}\right) = \frac{d}{dt}\left((t^2 - 3) \cdot t^{\frac{1}{2}}\right)$$

$$f'(t) = t^{\frac{1}{2}} \cdot \frac{d}{dt}(t^2 - 3) + (t^2 - 3) \cdot \frac{d}{dx}\left(t^{\frac{1}{2}}\right)$$

$$f'(t) = t^{\frac{1}{2}} \cdot [2 \cdot t^{2-1} - 0] + (t^2 - 3) \cdot \left[\frac{1}{2} \cdot t^{\frac{1}{2}-1}\right]$$

$$f'(t) = t^{\frac{1}{2}} \cdot [2 \cdot t] + (t^2 - 3) \cdot \left[\frac{1}{2} \cdot t^{-\frac{1}{2}}\right]$$

$$f'(t) = 2 \cdot t^{\frac{3}{2}} + \frac{1}{2} \cdot t^{\frac{3}{2}} - \frac{3}{2} \cdot t^{-\frac{1}{2}}$$

$$f'(t) = \frac{5}{2} \cdot t^{\frac{3}{2}} - \frac{3}{2} \cdot t^{-\frac{1}{2}} = \frac{5}{2} \cdot \sqrt{t^3} - \frac{3}{2 \cdot \sqrt{t}}$$

b) Use Algebra:

$$f'(t) = \frac{d}{dt}\left((t^2 - 3) \cdot t^{\frac{1}{2}}\right) = \frac{d}{dt}\left(t^{\frac{5}{2}} - 3 \cdot t^{\frac{1}{2}}\right)$$

$$f'(t) = \frac{5}{2} \cdot t^{\frac{3}{2}} - \frac{3}{2} \cdot t^{-\frac{1}{2}} = \frac{5}{2} \cdot \sqrt{t^3} - \frac{3}{2 \cdot \sqrt{t}}$$

Example Problem #12

Differentiate the function using the appropriate Derivative Rule.

$$f(x) = \frac{x^4 - 5x^3}{x^3}$$

Solution:

a) Use the Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - f\,g'}{g^2}$$

$$f'(x) = \frac{d}{dx}\left(\frac{x^4 - 5x^3}{x^3}\right)$$

$$f'(x) = \frac{x^3 \cdot \frac{d}{dx}(x^4 - 5x^3) - (x^4 - 5x^3) \cdot \frac{d}{dx}(x^3)}{(x^3)^2}$$

$$f'(x) = \frac{x^3 \cdot [4 \cdot x^3 - 5 \cdot (3 \cdot x^2)] - (x^4 - 5x^3) \cdot [3 \cdot x^2]}{x^6}$$

$$f'(x) = \frac{4 \cdot x^6 - 15 \cdot x^5 - 3 \cdot x^6 + 15 \cdot x^5}{x^6}$$

$$f'(x) = \frac{x^6}{x^6} = 1$$

b) We can use Algebra to first simplify the quotient:

$$f'(x) = \frac{d}{dx}\left(\frac{x^4 - 5x^3}{x^3}\right) = \frac{d}{dx}(x - 5) = 1$$

1.4 - EXERCISES

Find the derivatives of the following functions using the rules. If you notice that you can simplify the function first to avoid using the Product or Quotient Rules, then do that first.

1.	$f(x) = (x+1)(3x-5)$	2.	$y = (x^3 - 5x^2 - 1)(x^2 - 7x + 4)$
3.	$y = \left(\frac{x+1}{x-1}\right)$	4.	$f(x) = \left(\frac{2x^4}{4x+5}\right)$
5.	$y = \frac{(x+4)(x+3)}{(x^2-1)}$	6.	$f(x) = \sqrt[3]{x}(x^2 + 5x)$
7.	$y = (\sqrt{x} + 2)(\sqrt{x} - 6)$	8.	$y = \frac{3x^7 - 6x^5 + 9x^3 - 12x + 10}{3x^2}$
9.	$f(x) = \frac{3x+2}{5x-7}$	10.	$f(x) = \frac{1}{x+3} \cdot \frac{x^2}{4x-1}$

11. The cost of producing x kayaks is C(x) **hundreds of dollars**.

$$C(x) = 3x + 50$$

a) Find the average cost per kayak.

b) Evaluate the average cost when $x = 50$ and interpret the answer.

c) Find the marginal average cost.

d) Evaluate the marginal average cost when $x = 50$ and interpret the answer.

12.	A culture of bacteria grows in number according to the function (N(t)), where (t) is measured in hours. $$N(t) = 3000 \cdot \left(1 + \frac{4t}{t^2 + 100}\right)$$ a) Find the instantaneous rate of change of the number of bacteria. b) Find N′ (0) and N′ (10) and interpret your answers.
13.	A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by (P(x)), where x is the number of skateboards sold. $$P(x) = 30x - 0.3x^2 - 250$$ a) Find the exact profit from the sale of the thirteenth (13^{th}) skateboard. b) Find the marginal profit function and use it to estimate the profit from the sale of the thirteenth (13^{th}) skateboard. c) Find the marginal average profit function. d) Find the marginal average profit when $x = 13$ and interpret your answer.

Solutions:

1. $f'(x) = 6x - 2$

2. $\frac{dy}{dx} = (3x^2 - 10x)(x^2 - 7x + 4) + (2x - 7)(x^3 - 5x^2 - 1)$

3. $\frac{dy}{dx} = \frac{-2}{(x-1)^2}$

4. $f'(x) = \frac{24x^4+40x^3}{(4x+5)^2}$

5. $\frac{dy}{dx} = \frac{-7x^2-26x-7}{(x^2-1)^2}$

6. $f'(x) = \frac{7}{3} x^{4/3} + \frac{20}{3} x^{1/3}$

7. $\frac{dy}{dx} = 1 - \frac{2}{\sqrt{x}}$

8. $dy/dx = 5x^4 - 6x^2 + 3 + \frac{4}{x^2} - \frac{20}{3x^3}$

9. $f'(x) = \frac{-31}{(5x-7)^2}$

10. $f'(x) = \frac{-x^2}{(x+3)^2(4x-1)} + \frac{4x^2-2x}{(x+3)(4x-1)^2}$

11. a) $AC(x) = \frac{3x+50}{x}$,

    b) $AC(50) = 4$, when 50 kayaks are produced, the average cost per kayak is $400/kayak.

    c) $MAC(x) = -50/x^2$, d) $MAC(50) = -0.02$, after 50 kayaks are produced the average cost per kayak decreases by 0.02 hundreds of dollars (= 2 dollars) per each additional kayak produced.

12. a) $N'(t) = \frac{-12000t^2+1200000}{(t^2+100)^2}$

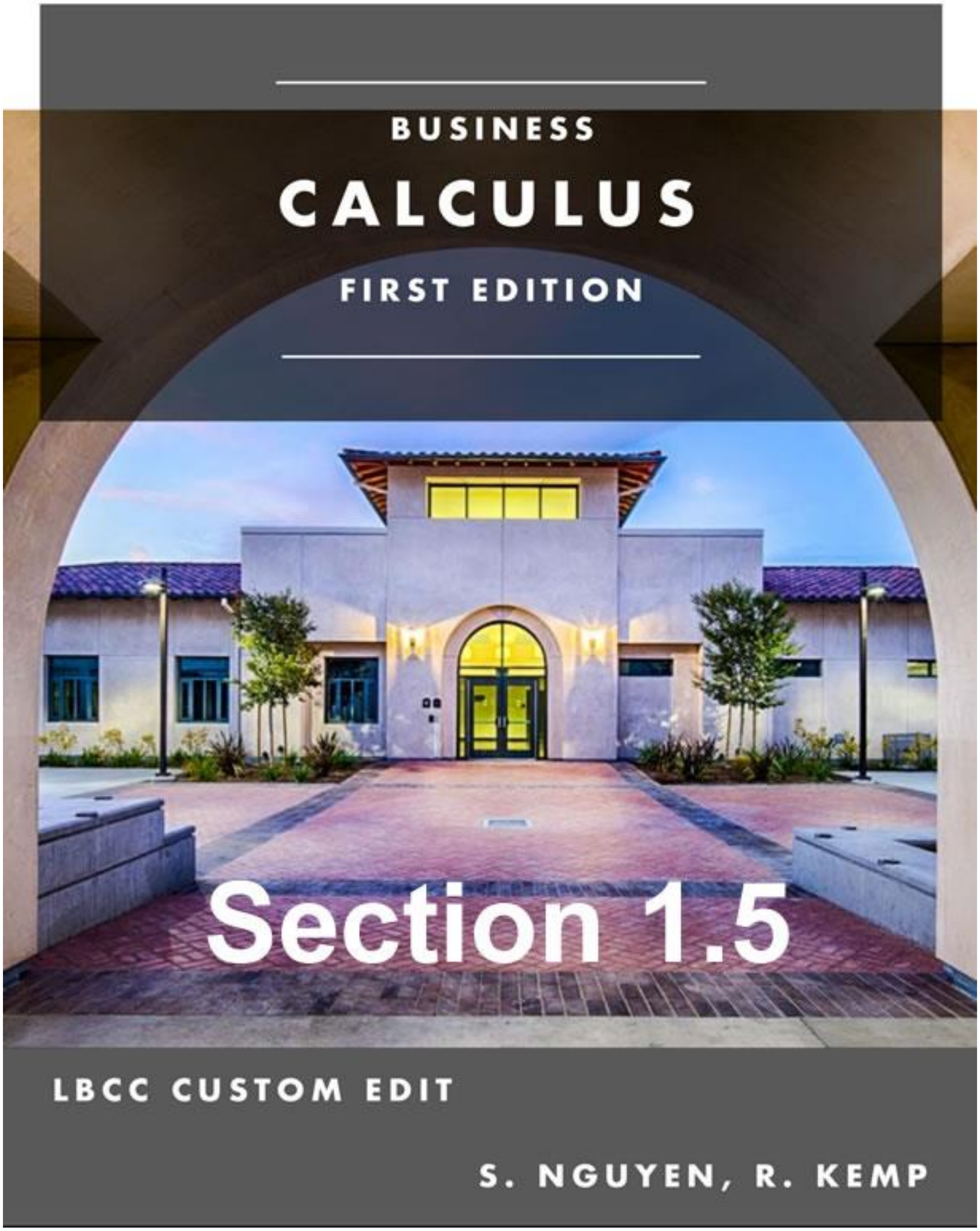
b) $N'(0) = 120, N'(10) = 0$. Initially, at time 0, the bacteria multiplies at a rate of 120 new bacteria per hour. After 10 hours, the culture of bacteria stopped growing.

13. a) $P(13) - P(12) = 22.5$ dollars for the 13th skateboard.

b) $MP(x) = 30 - 0.6x, \ MP(13) = 22.2$ dollars/skateboard

c) $MAP(x) = -0.3 + \frac{250}{x^2}$

d) $MAP(13) = 1.18$. After 13 skateboards are sold the average profit per additional skateboard sold increases by \$1.18/skateboard.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1.5 - HIGHER ORDER DERIVATIVES

Since the derivative of a function is a function as well, we can in turn find its derivative. This would now be the second derivative of the original function.

Second Derivative

Let the **Initial Value Function**, $y = f(x)$

The **second derivative of** ($f(x)$) is the derivative of ($y' = f'(x) = \frac{d}{dx}f(x)$).

Using prime notation, this is ($f''(x)$) or (y'').

You can read this aloud as "y double prime."

Using Leibniz notation, the second derivative is written: ($\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}f(x)$).

$$y'' = f''(x) = \frac{d}{dx}f'(x) = \frac{d^2}{dx^2}f(x)$$

This is read aloud as "the second derivative of f ".

Higher Order Derivatives

The **third derivative of** ($f(x)$) is the derivative of ($y'' = f''(x)$).

Using prime notation, this is ($f'''(x)$) or (y''').

You can read this aloud as "y triple prime."

Using Leibniz notation, the third derivative is written: ($\frac{d^3y}{dx^3}$ or $\frac{d^3}{dx^3}f(x)$)

This is read aloud as "the third derivative of f ".

We can continue on, but it would be difficult to start counting the primes after 3, so the notation for the **nth derivative** from the 4th derivative on is:

$$f^{(n)}(x) = y^{(n)} \quad or \quad \frac{d^ny}{dx^n} \quad or \quad \frac{d^n}{dx^n}f(x)$$

Example Problem #1

Given the function $f(x)$, find $f''(x)$

$$f(x) = 2x^3 + 5x^2 - 7x + 5$$

Solution:

We must find the first derivative first:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(2x^3 + 5x^2 - 7x + 5)$$

$$f'(x) = 6x^2 + 10x - 7$$

Then, the second derivative will be found by taking the derivative of $(f'(x))$

$$f''(x) = \frac{d^2}{dx^2}f(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}(6x^2 + 10x - 7)$$

$$f''(x) = 12x + 10$$

Example Problem #2

Given the function $f(x)$, find $(\frac{d^3}{dx^3}f(x))$

$$f(x) = 4\sqrt[4]{x^3}$$

Solution:

First, we need to rewrite the radical in the function as a power:

$$f(x) = 4 \cdot x^{\frac{3}{4}}$$

Find the first derivative:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[4 \cdot x^{\frac{3}{4}}\right] = 4 \cdot \left(\frac{3}{4}x^{-\frac{1}{4}}\right) = 3x^{-\frac{1}{4}}$$

Find the second derivative:

$$\frac{d^2}{dx^2}f(x) = \frac{d}{dx}\left[\frac{d}{dx}f(x)\right] = \frac{d}{dx}\left[3x^{-\frac{1}{4}}\right]$$

$$\frac{d^2}{dx^2}f(x) = 3 \cdot \left(-\frac{1}{4}x^{\left(-\frac{1}{4}-1\right)}\right) = -\frac{3}{4}x^{-\frac{5}{4}}$$

Find the third derivative:

$$\frac{d^3}{dx^3}f(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2}f(x)\right] = \frac{d}{dx}\left[-\frac{3}{4}x^{-\frac{5}{4}}\right]$$

$$\frac{d^3}{dx^3}f(x) = -\frac{3}{4}\left(-\frac{5}{4}x^{\left(-\frac{5}{4}-1\right)}\right) = \frac{15}{16}x^{-\frac{9}{4}}$$

Example Problem #3

Given the function $f(x)$, find ($f^4(2)$):

$$f(x) = \frac{2}{x^2}$$

Solution:

What we need to find is the fourth derivative of *f* (x) and then evaluate it at x = 2.

To use the power rule, we need to rewrite the function ($f(x)$)

$$f(x) = 2x^{-2}$$

Find the first derivative:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(2x^{-2}) = 2\left(-2x^{(-3-1)}\right) = -4x^{-3} = -\frac{4}{x^3}$$

Find the second derivative:

$$f''(x) = \frac{d^2}{dx^2}f(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}(-4x^{-3})$$

$$f''(x) = -4\left(-3x^{(-3\,-1)}\right) = 12x^{-4} = \frac{12}{x^4}$$

Find the third derivative:

$$f'''(x) = \frac{d^3}{dx^3}f(x) = \frac{d}{dx}f''(x) = \frac{d}{dx}(12x^{-4})$$

$$f'''(x) = 12\left(-4x^{(-4\,-1)}\right) = -48x^{-5} = -\frac{48}{x^5}$$

Find the fourth derivative:

$$f^4(x) = \frac{d^4}{dx^4}f(x) = \frac{d}{dx}f'''(x) = \frac{d}{dx}(-48x^{-5})$$

$$f^4(x) = -48\left(-5x^{(-5\,-\,1)}\right) = 240x^{-6} = \frac{240}{x^6}$$

Next, evaluate it at (x = 2)

$$f^4(2) = \frac{240}{(2^6)} = \frac{240}{64} = 3.75$$

Velocity and Acceleration

One application from physics of derivatives is the velocity and acceleration functions.

For example, the derivative of a position function is the rate of change of position, or velocity.

The derivative of velocity is the rate of change of velocity, which is acceleration.

Definition:

Let [$s(t)$] be a function giving the **position** of an object at time [t].

$$s(t)$$

The **velocity** of the object at time t is given by [$v(t) = s'(t)$]

$$v(t) = s'(t) = \frac{d}{dt}s(t)$$

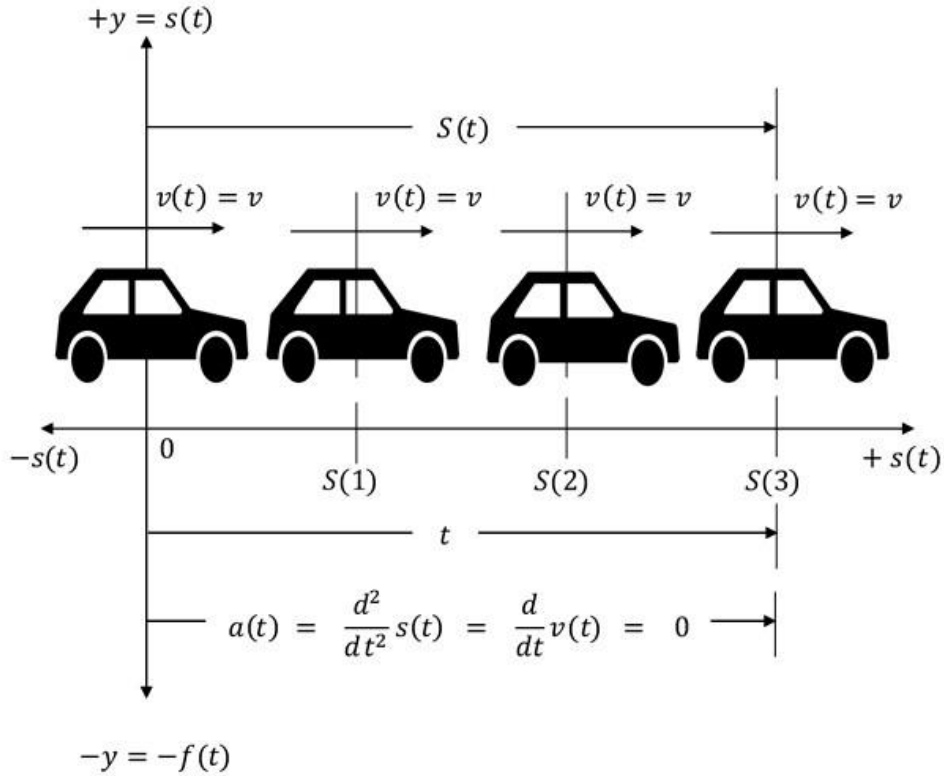
The **speed** of the object at time t is given by [$|v(t)|$].

$$|v(t)| = |s'(t)| = \left|\frac{d}{dt}s(t)\right|$$

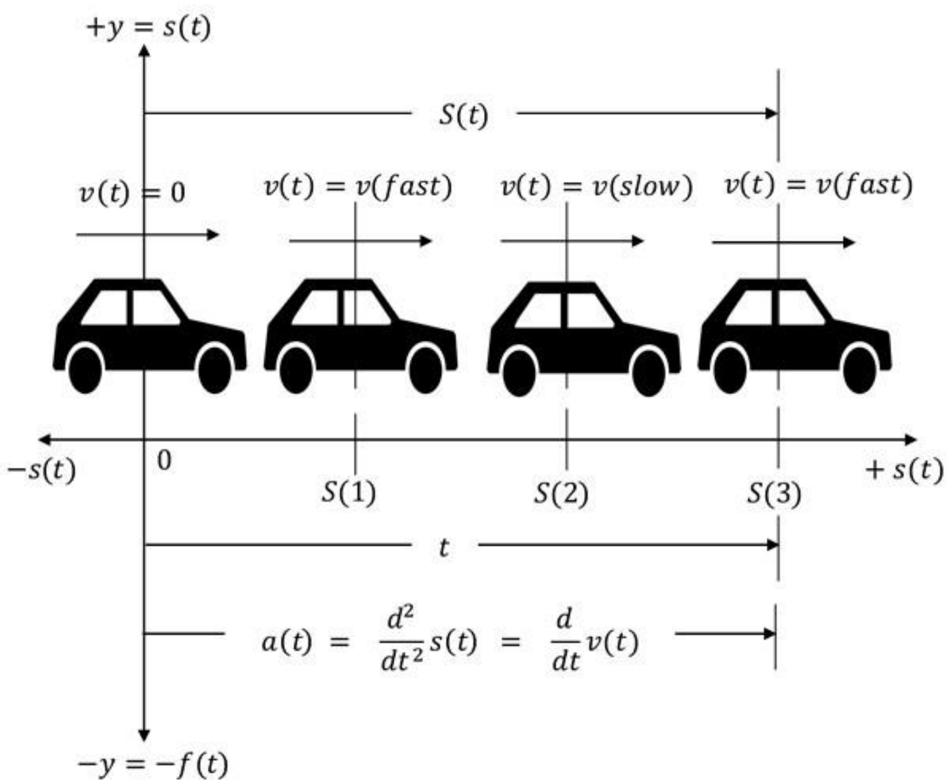
The **acceleration** of the object at t is given by [$a(t) = v'(t) = s''(t)$]

$$a(t) = v'(t) = s''(t)$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}s(t)$$



$+y = s(t)$
$S(t)$
$v(t) = v$
$v(t) = v$
$v(t) = v$
$v(t) = v$
$-s(t)$
0
$S(1)$
$S(2)$
$S(3)$
$+s(t)$
t
$a(t) = \frac{d^2}{dt^2}s(t) = \frac{d}{dt}v(t) = 0$
$-y = -f(t)$



$+y = s(t)$
$S(t)$
$v(t) = 0$
$v(t) = v(fast)$
$v(t) = v(slow)$
$v(t) = v(fast)$
$-s(t)$
0
$S(1)$
$S(2)$
$S(3)$
$+s(t)$
t
$a(t) = \frac{d^2}{dt^2}s(t) = \frac{d}{dt}v(t)$
$-y = -f(t)$

Example Problem #4

Interpreting the Relationship between the velocity ($v(t)$) and acceleration ($a(t)$)

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by ($s(t)$).

$$s(t) = t^3 - 4t + 2$$

Find ($v(1)$) and ($a(1)$) and use these values to answer the following questions.

a) Is the particle moving from left to right or from right to left at time (t = 1)?

b) Is the particle speeding up or slowing down at time (t = 1)?

Solution:

Begin by finding Velocity ($v(t)$) and acceleration ($a(t)$).

Velocity:

$$v(t) = \frac{d}{dt}s(t) = \frac{d}{dt}(t^3 - 4t + 2) = 3t^2 - 4$$

$$v(1) = 3(1)^2 - 4 = 3 - 4 = -1$$

Acceleration:

$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}s(t) = \frac{d}{dt}(3t^2 - 4) = 6t$$

$$a(1) = 6(1) = 6$$

Evaluating these functions at (t = 1), we obtain ($v(1) = -1$) and ($a(1) = 6$).

a) Because ($v(1) < 0$) the velocity of the particle is negative, and is moving from right to left.

b) Because ($v(1) < 0$) and ($a(1) > 0$), velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing |v(t)| to decrease. The particle is slowing down.

Example Problem #5

The height (feet) of a particle at time (t) seconds is ($f(t)$).

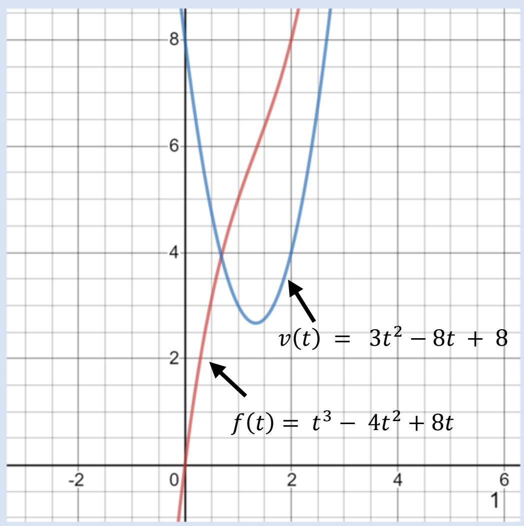
$$f(t) = t^3 - 4t^2 + 8t$$

Find the height, velocity and acceleration of the particle when (t = 0 sec, t= 1 sec, and t = 2 sec).

Solution:

$$f(t) = t^3 - 4t^2 + 8t$$

So, the different heights are: f(0) = 0 feet, f(1) = 5 feet, f(2) = 8 feet.



In the graph above, the red curve describes the position of the particle and the blue curve describes the velocity of the particle.

The velocity at time (t) is:

$$v(t) = f'(t) = \frac{d}{dt}(t^3 - 4t^2 + 8t) = 3t^2 - 8t + 8$$

So, the different velocities are: v(0) = 8 ft/s , v(1) = 3 ft/s, v(2) = 4 ft/s.

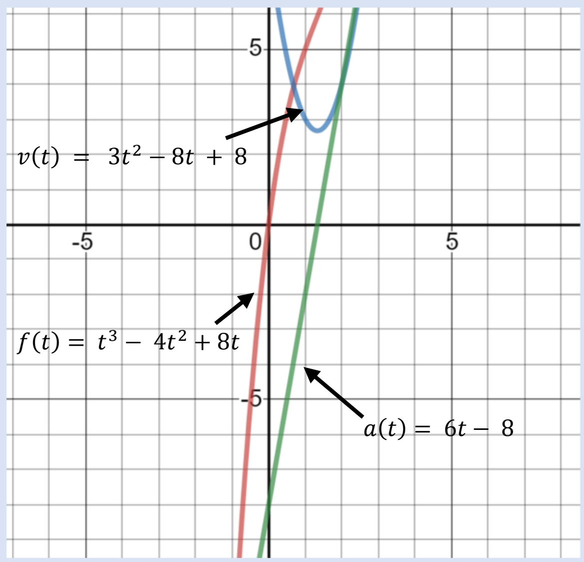
At each of these times the velocity is positive and the particle is moving upward, increasing in height.

Example Problem #5 – Cont'd

The **acceleration** at time (t) is:

$$a(t) = v'(t) = f''(t) = \frac{d}{dt}(3t^2 - 8t + 8) = 6t - 8$$

The different accelerations are: a(0) = –8 ft/s^2, a(1) = –2 ft/s^2 , a(2) = 4 ft/s^2.



In the graph above, the red curve describes the position of the particle and the blue curve describes the velocity of the particle, and the green curve describes the acceleration of the particle.

At time (t = 0 sec) and (t = 1 sec), the acceleration is negative, so the particle's velocity would be decreasing at those points - the particle was slowing down.

At time (t = 2 sec), the velocity is positive, so the particle was increasing in speed.

Example Problem #6

A supply function (the amount of items a producer is willing to sell at a certain price) for a certain product is given by ($S(p)$) many units, where (p) is the price of the product in dollars.

$$S(p) = 0.05p^3 + 2p^2 - 4p + 20$$

Find:

a) The number of units being sold when the product is $4.

b) The rate of change in the number of units sold when the price is $4.

c) The rate of change of the rate of change in the number of units being sold when the price is $4.

Solution:

a) The number of units being sold when the product is $4

$$S(4) = 0.05(4)^3 + 2(4)^2 - 4(4) + 20$$

$$S(4) = 0.05(64) + 32 - 16 + 20 = 39.2\ units$$

So, the supplier is willing to sell about 39 units when the price is $4.

b) The rate of change in the number of units sold when the price is $4

$$S'(p) = \frac{d}{dp}S(p) = \frac{d}{dp}(0.05p^3 + 2p^2 - 4p + 20)$$

$$S'(p) = 0.05(3p^2) + 2(2p) - 4$$

$$S'(p) = 0.15p^2 + 4p - 4$$

Next, when the price is (p = $4).

$$S'(4) = 0.15(4)^2 + 4(4) - 4 = 0.15(16) + 16 - 4$$

$$S'(4) = 14.4\ \ units/dollar$$

This means that the supplier is willing to sell about 14 more units for every extra dollar increase in price when the price is at $4

Example Problem #6 – Cont'd

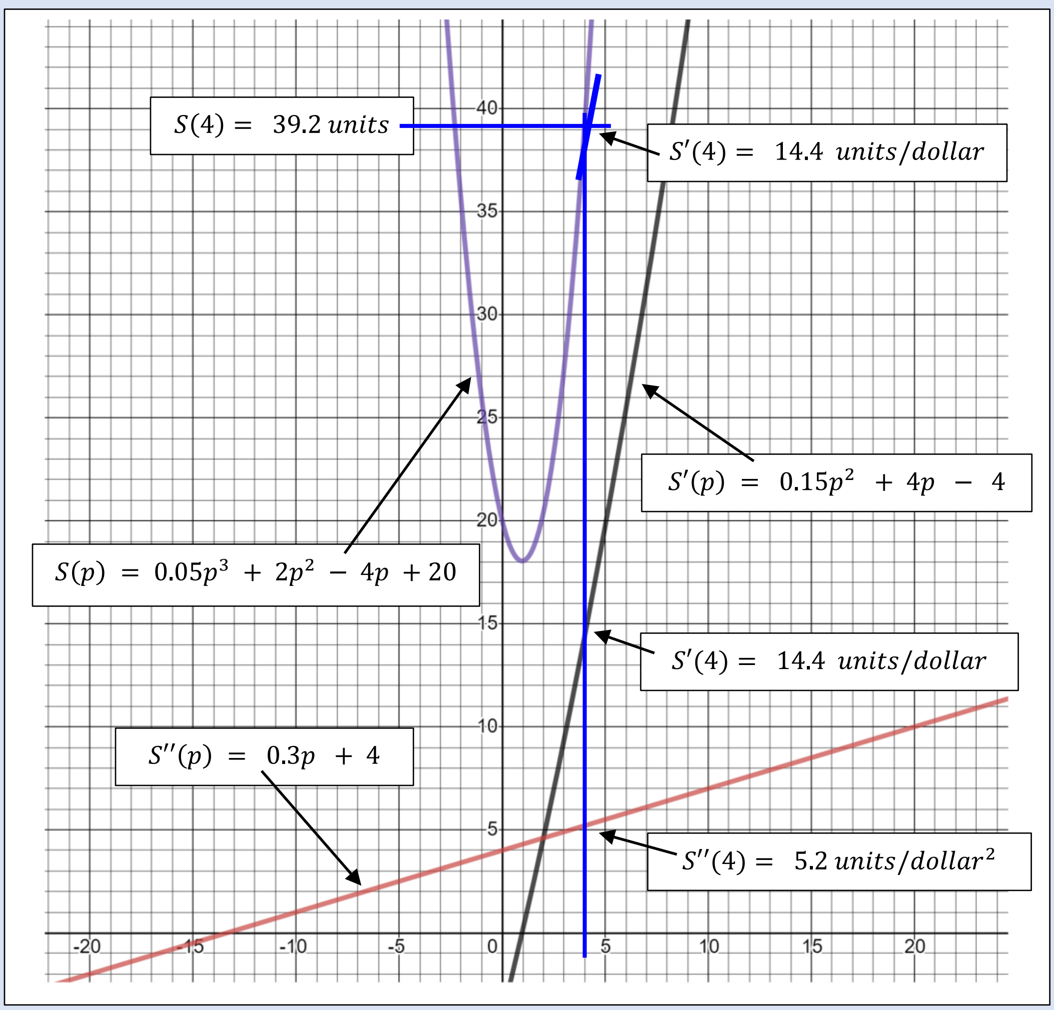
c) The rate of change of the rate of change in the number of units being sold when the price is $4

$$S''(p) = \frac{d}{dp}S'(p) = \frac{d^2}{dp^2}S(p) = \frac{d}{dp}(0.15p^2 + 4p - 4)$$

$$S''(p) = 0.15(2p) + 4 = 0.3p + 4$$

$$S''(4) = 5.2 \ \ units/dollar^2$$

When the price is $4 the rate of at which the number of items sold per dollar is changing is increasing by 5 units/dollar per extra dollar increase in price.



1.5 - EXERCISES

Given the following functions, find the second derivative $\frac{d^2y}{dx^2}$			
1.	$y = 4x^2 - 7x + 25$	2.	$y = (2x - 5)(4x + 7)$
3.	$y = 3\sqrt[3]{x^2}$	4.	$y = 4x^7 + \frac{2}{x} + 7$
5.	$y = \frac{x^5}{10} + \frac{10}{x^5}$	6.	$y = \frac{x+1}{x-1}$
7.	Given the function $f(x) = 7x^7 - 5x^5 + 4x^4 - 2x^2$ find $f^{(4)}(1)$.	8.	Given $y = 3x^4 + \sqrt{x}$, find $\left.\frac{d^2y}{dx^2}\right\|_{x=4}$
9.	Given $f(x) = \frac{1}{x^3}$, find $f'''(2)$	10.	Given $y = (3x^2 - 5)(3x^2 + 5)$ find $\left.\frac{d^3y}{dx^3}\right\|_{x=1}$

11.	A population grows from an initial size of 100,000, according to the law of $P(x) = 100{,}000(1 + 0.2x + x^2)$ where x is the number of years. Find the acceleration in the size of the population.

12.	A potato is launched vertically upward with an initial velocity of 100ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels after t seconds is given by $$s(t) = -16t^2 + 100t + 85.$$ a) Find the velocity of the potato after 0.5 s and 2.5 s. b) Find the acceleration of the potato at 0.5 s and 2.5 s.
13.	A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels after t seconds is given by $$s(t) = -16t^2 + 560t.$$ a) Find the velocity of the rocket 3 seconds after being fired. b) Find the acceleration of the rocket 3 seconds after being fired.
14.	A town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function: $$P(t) = -13t^3 + 64t + 3000,$$ where t is measured in years from now. a. Find the rate of change function $P'(t)$ of the population function. b. Find $P'(1)$, and $P'(2)$. Interpret what the results mean for the town. c. Find $P''(1)$, and $P''(2)$. Interpret what the results mean for the town's population.

Solutions:

1. $\frac{d^2y}{dx^2} = 8$

2. $\frac{d^2y}{dx^2} = 16$

3. $\frac{d^2y}{dx^2} = -\frac{2}{3\sqrt[3]{x^4}}$

4. $\frac{d^2y}{dx^2} = 168x^5 + \frac{4}{x^3}$

5. $\frac{d^2y}{dx^2} = 2x^3 + \frac{300}{x^7}$

6. $\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3}$

7. 5376

8. $575\frac{31}{32}$

9. $-15/16$

10. 216

11. $200{,}000$ people/ year/ year

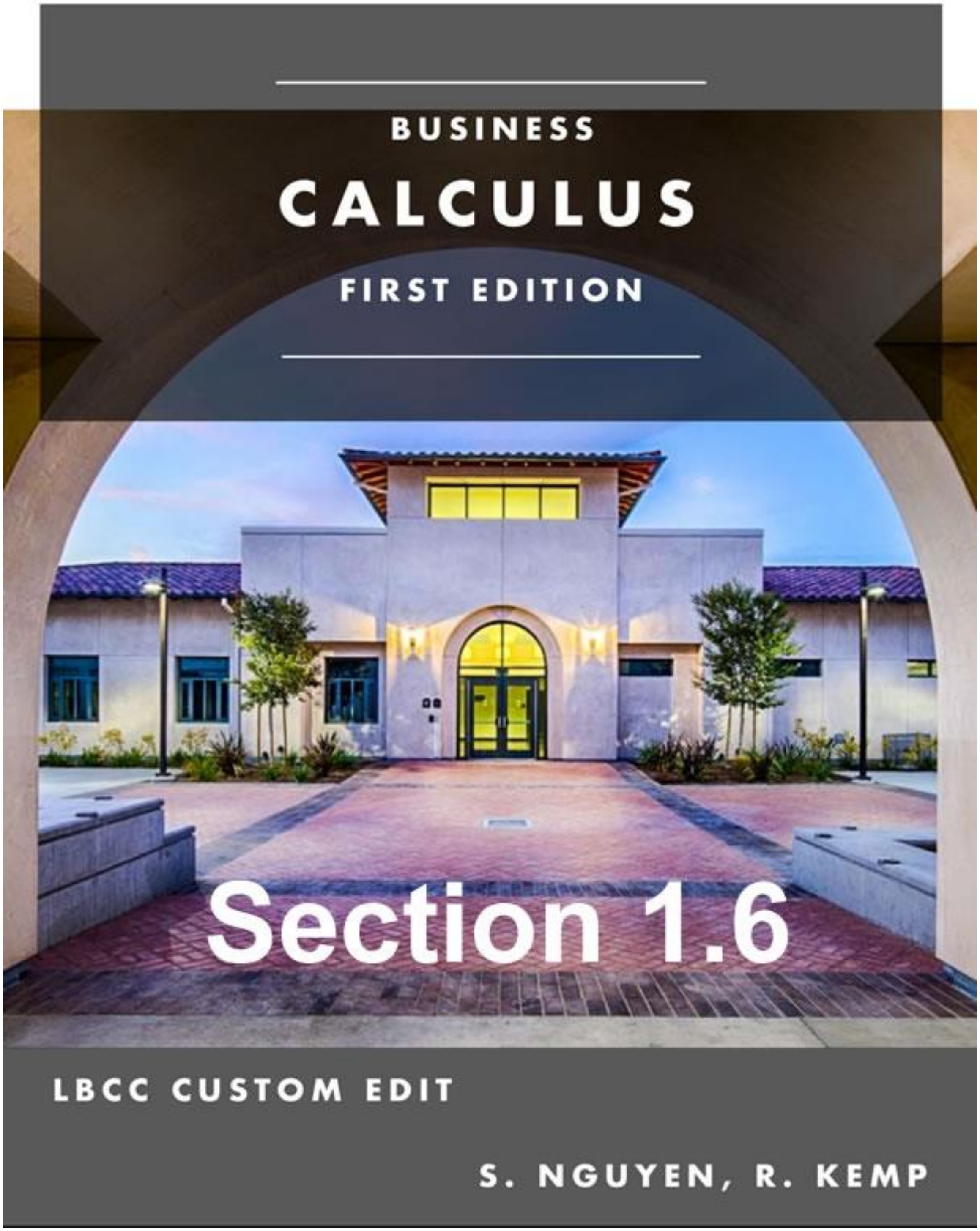
12. a) $v(0.5) = 84ft/s,\ v(2.5) = 20ft/s$
    b) $a(0.5) = -32\ ft/s^2,\ a(2.5) = -32\ ft/s^2$

13. a) $v(3) = 464ft/s$
    b) $a(3) = -32ft/s^2$

14. a) $P'(t) = -39t^2 + 64$

b) $P'(1) = 25, P'(2) = -92$. The population of the town will increase by 25 thousand people/year after one year from now, but it will be decreasing by 92 thousand people/year after 2 years from now.

c) $P"(1) = -78, P"(2) = -156$. In one year from now the population growth will be deaccelerated by 78 thousand people/year/year and in 2 years it will deaccelerate even more by 156 thousand people/year/year.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1.6 - THE CHAIN RULE

There is one more type of complicated function that we will want to know how to differentiate: **The composition of functions**. We looked at how to take derivatives of one type of function at a time, but what if the functions are composed?

The **Chain Rule** will let us find the derivative of a composition.

Now suppose we want to find the derivative of ($y = (4x^3 + 15x)^{10}$).

We **could** write it as a product with 10 factors and use the product rule, or we **could** multiply it out! But I wouldn't want to do that, would you?

We need an easier way, a rule that will handle a composition like this.

The Chain Rule is a little complicated, but it saves us the much more complicated algebra of multiplying something like this out.

It will also handle compositions where it wouldn't be possible to be "multiplied out" as we will see in Chapter 3.

The Chain Rule is the most common place for students to make mistakes. Part of the reason is that the notation takes a little getting used to.

And part of the reason is that students often forget to use it when they should. When should you use the Chain Rule? Almost every time you take a derivative.

Derivative Rules: Chain Rule

In what follows, (f) and (g) are differentiable functions with ($y = f(u)$) and ($u = g(x)$)

(f is a function of (u), and (u) in turns is a function of (x)

$$y = f(g(x))$$

Chain Rule (Leibnitz Notation):

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \left[\frac{d}{du} f(u)\right] \cdot \left[\frac{d}{dx} g(x)\right] = f'(u) \cdot g'(x)$$

Notice that the (du) seem to cancel. This is one advantage of the Leibniz notation; it can remind you of how the chain rule chains together.

Chain Rule (Using Prime Notation):

$$y' = \frac{dy}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

The Generalized Power Rule/Chain Rule:

$$y' = \frac{dy}{dx} = \frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot \left[\frac{d}{dx} g(x)\right]$$

$$y' = \frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

Chain Rule (in words):

The derivative of a composition is the derivative of the outside, with the inside staying the same, TIMES the derivative of what's inside.

Example Problem #1

Find the derivative of:

$$y = (4x^3 + 15x)^{10}$$

Solution:

This function is made of the composition of a power function ($y = f(x)$), (on the outside) and a polynomial function ($g(x)$), (on the inside)

$$y = f(x) = (x)^{10} \qquad g(x) = 4x^3 + 15x$$

$$y = f(g(x))$$

Following the Chain Rule, we must first take the derivative of the outside function and leave the inside function alone, then multiplied by the derivative of the inside function alone:

$$y' = \frac{dy}{dx} = f'(g(x)) \cdot g'(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$y' = \frac{dy}{dx} = \frac{d}{dx}(4x^3 + 15x)^{10}$$

$$y' = \frac{dy}{dx} = 10(4x^3 + 15x)^{10-1}\left[\frac{d}{dx}(4x^3 + 15x)\right]$$

$$y' = \frac{dy}{dx} = 10(4x^3 + 15x)^9 \cdot (12x^2 + 15)$$

You can leave the answer as it. Do not try to multiply this one out!

Example Problem #2

Find the derivative of the function:

$$f(x) = \sqrt[3]{x^4 + 7x^2 - 9x + 18}$$

Solution:

Here we are also dealing with the composition of two functions.

The inside function is the polynomial ($x^4 + 7x^2 - 9x + 18$) and around it wraps the outside radical function.

This function is made of the composition of a power function ($h(x)$), (on the outside) and a polynomial function ($g(x)$), (on the inside)

$$h(x) = (x)^{\frac{1}{3}} \qquad g(x) = x^4 + 7x^2 - 9x + 18$$

$$f(x) = h(g(x))$$

First, we must rewrite the radical as an exponent, so we use the power rule.

$$f(x) = (x^4 + 7x^2 - 9x + 18)^{1/3}$$

Next, take the derivative of the function:

$$f'(x) = \frac{d}{dx}f(x) = h'(g(x)) \cdot g'(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(x^4 + 7x^2 - 9x + 18)^{1/3}$$

$$f'(x) = \frac{1}{3}(x^4 + 7x^2 - 9x + 18)^{(1/3 - 1)}\left[\frac{d}{dx}(x^4 + 7x^2 - 9x + 18)\right]$$

$$f'(x) = \frac{1}{3}(x^4 + 7x^2 - 9x + 18)^{-2/3} \cdot (4x^3 + 14x - 9)$$

$$f'(x) = \frac{4x^3 + 14x - 9}{3\sqrt[3]{(x^4 + 7x^2 - 9x + 18)^2}}$$

Example Problem #3

Find the equation of a line tangent to the graph of:

$$h(x) = \left(\frac{1}{3x-5}\right)^2 = \frac{1}{(3x-5)^2} \qquad at \qquad x = 2$$

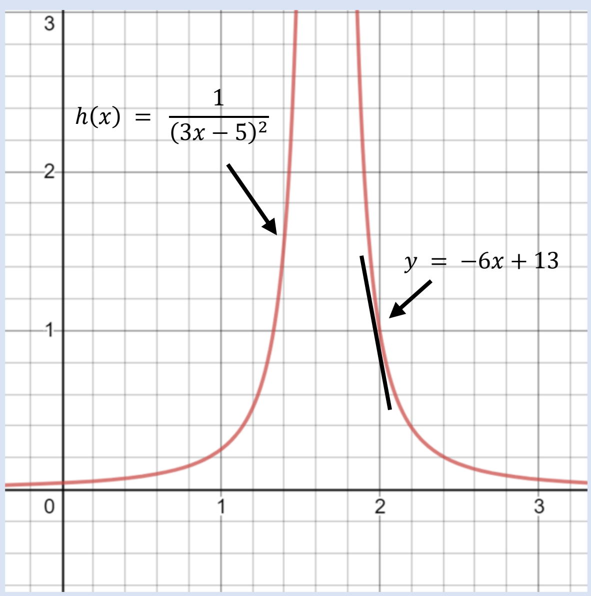
Solution:

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2.

To find the y-coordinate, substitute 2 into ($h(x)$).

$$y = h(2) = \frac{1}{(3(2)-5)^2} = 1$$

Since ($h(2) = 1$), the point that is tangent to the line is ($(x, y) = (2, 1)$).



For the slope, we need ($m = h'(2)$).

To find ($h'(x)$), first we rewrite ($h(x)$) and apply the chain rule to obtain

$$h(x) = (3x-5)^{-2}$$

Example Problem #3 – Cont'd

To find ($h'(x)$), apply the chain rule to obtain

$$h'(x) = \frac{d}{dx}(3x-5)^{-2}$$

$$h'(x) = -2(3x-5)^{(-2-1)}\left[\frac{d}{dx}(3x-5)\right]$$

$$h'(x) = -2(3x-5)^{-3}(3)$$

$$h'(x) = -6(3x-5)^{-3} = \frac{-6}{(3x-5)^3}$$

By substituting, we have the slope;

$$m = h'(2) = \frac{-6}{(3(2) - 5)^3} = -6$$

Therefore, using the point-slope formula, the line has equation:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -6(x - 2)$$

Rewriting, the equation of the tangent line is given by:

$$y = -6x + 13$$

Example Problem #4

Find the derivative of the function:

$$f(x) = \left(\frac{1}{x^3 - 7x}\right)^3$$

Solution:

First, we must rewrite the radical as an exponent, so we use the power rule.

$$f(x) = (x^3 - 7x)^{-3}$$

Next, take the derivative of the function:

$$f'(x) = \frac{d}{dx}f(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$f'(x) = \left[-3(x^3 - 7x)^{(-3-1)}\right] \cdot \frac{d}{dx}(x^3 - 7x)$$

$$f'(x) = -3(x^3 - 7x)^{-4} \cdot (3x^2 - 7)$$

$$f'(x) = \frac{-3 \cdot (3x^2 - 7)}{(x^3 - 7x)^4} = \frac{-9x^2 + 21}{(x^3 - 7x)^4}$$

Example Problem #5

Find the derivative of the function:

$$f(x) = \left(\frac{x + 2}{x - 5}\right)^6$$

Solution:

Next, take the derivative of the function:

$$f'(x) = \frac{d}{dx}f(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{x + 2}{x - 5}\right)^6$$

$$f'(x) = \left[6 \cdot \left(\frac{x + 2}{x - 5}\right)^{(6-1)}\right] \cdot \frac{d}{dx}\left(\frac{x + 2}{x - 5}\right)$$

$$f'(x) = 6 \cdot \left(\frac{x + 2}{x - 5}\right)^{6-1} \cdot \left[\frac{(x-5) \cdot \frac{d}{dx}(x+2) - (x+2) \cdot \frac{d}{dx}(x-5)}{(x - 5)^2}\right]$$

$$f'(x) = 6 \cdot \left(\frac{x + 2}{x - 5}\right)^5 \cdot \left[\frac{(x-5) \cdot (1) - (x+2) \cdot (1)}{(x - 5)^2}\right]$$

$$f'(x) = 6 \cdot \left(\frac{x + 2}{x - 5}\right)^5 \cdot \left[\frac{x - 5 - x - 2}{(x - 5)^2}\right]$$

$$f'(x) = 6 \cdot \left(\frac{x + 2}{x - 5}\right)^5 \cdot \left[\frac{-7}{(x - 5)^2}\right]$$

$$f'(x) = -42 \cdot \left[\frac{(x + 2)^5}{(x - 5)^7}\right]$$

Example Problem #6

Find the derivative of the function:

$$f(x) = (4x-2)^5(x+6)^4$$

Solution:

Next, take the derivative of the function:

$$f'(x) = \frac{d}{dx}f(x) = n\cdot[g(x)]^{n-1}\cdot g'(x)$$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}[(4x-2)^5(x+6)^4]$$

$$f'(x) = (x+6)^4\cdot\frac{d}{dx}(4x-2)^5 + (4x-2)^5\cdot\frac{d}{dx}(x+6)^4$$

$$f'(x) = (x+6)^4\cdot\left[5\cdot(4x-2)^{(5-1)}\cdot\frac{d}{dx}(4x-2)\right] + (4x-2)^5\cdot\left[4\cdot(x+6)^{(4-1)}\cdot\frac{d}{dx}(x+6)\right]$$

$$f'(x) = (x+6)^4\cdot[5\cdot(4x-2)^4\cdot(4)] + (4x-2)^5\cdot[4\cdot(x+6)^3\cdot(1)]$$

$$f'(x) = 20(x+6)^4\cdot(4x-2)^4 + 4(4x-2)^5\cdot(x+6)^3$$

$$f'(x) = 4(x+6)^3\cdot(4x-2)^4\cdot[5(x+6) + (4x-2)]$$

$$f'(x) = 4\cdot(x+6)^3\cdot(4x-2)^4\cdot[5x + 30 + 4x-2]$$

$$f'(x) = 4\cdot(x+6)^3\cdot(4x-2)^4\cdot(9x + 28)$$

Example Problem #7

Find the derivative of the function:

$$f(x) = [x^3 - (x^3 + 5)^4]^7$$

Solution:

Next, take the derivative of the function:

$$f'(x) = \frac{d}{dx} f(x) = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}[x^3 - (x^3 + 5)^4]^7$$

$$f'(x) = \left(7 \cdot [x^3 - (x^3 + 5)^4]^{(7-1)}\right) \cdot \frac{d}{dx}[x^3 - (x^3 + 5)^4]$$

$$f'(x) = (7 \cdot [x^3 - (x^3 + 5)^4]^6) \cdot \left[\frac{d}{dx} x^3 - \frac{d}{dx}(x^3 + 5)^4\right]$$

$$f'(x) = 7 \cdot [x^3 - (x^3 + 5)^4]^6 \cdot [3x^2 - 4 \cdot (x^3 + 5)^3 \cdot (3x^2)]$$

Or

$$f'(x) = 7 \cdot [x^3 - (x^3 + 5)^4]^6 \cdot [3x^2 - 12x^2 \cdot (x^3 + 5)^3]$$

$$f'(x) = 21x^2 \cdot [x^3 - (x^3 + 5)^4]^6 \cdot [1 - 4 \cdot (x^3 + 5)^3]$$

Or

$$f'(x) = 7 \cdot [x^3 - (x^3 + 5)^4]^6 \cdot [21x^2 - 84x^2 \cdot (x^3 + 5)^3]$$

What if the Derivative Doesn't Exist?

A function is called **differentiable** at a point if its derivative exists at that point. We've been acting as if derivatives exist everywhere for every function.

This is true for most of the functions that you will run into in this class. But there are some common places where the derivative doesn't exist.

Remember that the derivative is the **slope** of the **tangent line** to the curve.

Where can a slope not exist? If the **tangent line is vertical**, the derivative will not exist, because the slope of a tangent line is undefined.

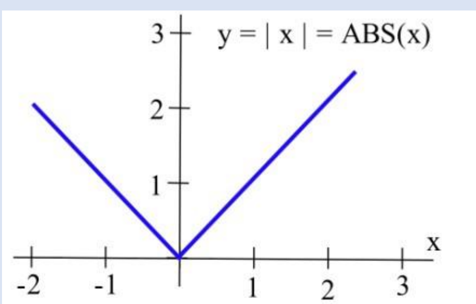
Where can a tangent line not exist? If there is a **sharp corner** (cusp) in the graph, the derivative will not exist at that point because there is no well-defined tangent line (a teetering tangent, if you will).

If the function is **not continuous** at a point, i.e., there is a jump in the graph, then tangent line will be different on either side and the derivative can't exist, or if a point is missing from the graph, then we cannot draw a tangent line at that point.

Example Problem #8

Show that $f(x) = |x|$ is not differentiable at x = 0.

Solution:



$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Left Side

$$f(x) = \frac{d}{dx} f(x) = \frac{d}{dx}(-x) = -1$$

Right Side

$$f(x) = \frac{d}{dx} f(x) = \frac{d}{dx}(x) = 1$$

Here is the graph of the absolute value function. As you can see the slope on the left side of x = 0 is equal to -1. On the right side of the graph from x = 0, the slope equals to 1.

There is no well-defined tangent line at the sharp corner x = 0, so the function does not have a derivative at that point, i.e. is not differentiable at x= 0.

Example Problem #9

Show that $f(x)$ is not differentiable at x = 0.

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

Solution:

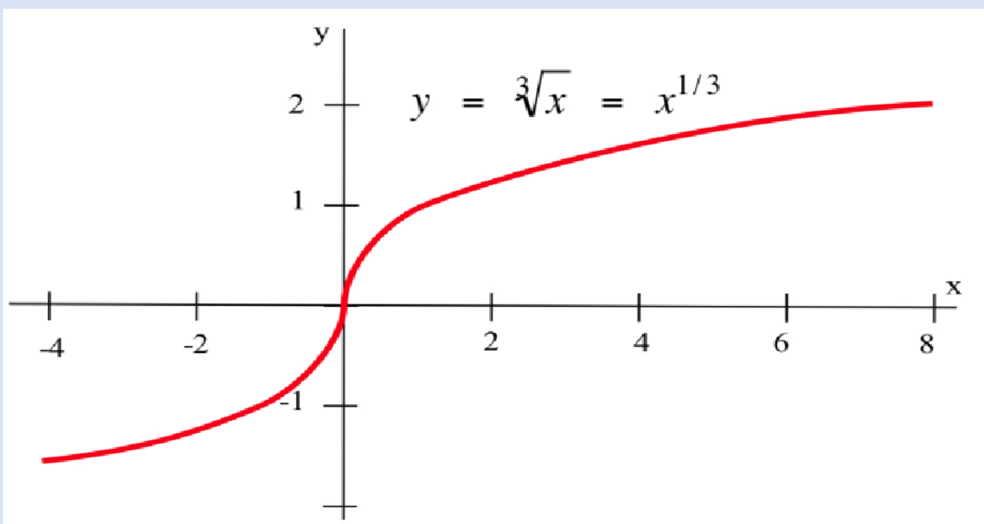
Finding the derivative:

$$f'(x) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{((1/3)-1)} = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

Since, ($f'(0)$) is undefined, so the derivative at (x = 0) does not exist, hence the function is not differentiable at (x = 0).

$$f'(0) = \frac{1}{3\sqrt[3]{0^2}} = \frac{1}{0} = Undefined$$

From the graph, we see that the function ($f(x)$) has a vertical slope at (x = 0), which means the slope of the tangent line at this point is undefined, which is why we found that the derivative at x = o does not exist.



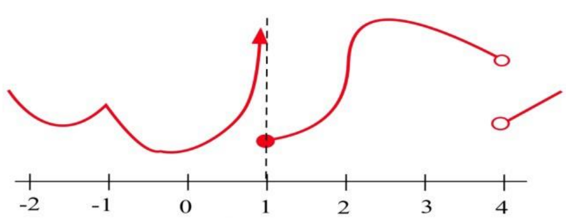
1.6 - EXERCISES

Find the derivative of the following functions using a combination of all the rules we studied.			
1.	$f(x) = (2x - 8)^5$	2.	$f(x) = (6x - x^2)^{10}$
3.	$f(x) = x(3x + 7)^5$	4.	$f(x) = (2x + 3)^6(x - 2)^4$
5.	$f(x) = \sqrt[3]{x^2 + 6x - 1}$	6.	$y = \frac{x+2}{(x-3)^5}$
7.	$f(x) = \frac{1}{(x^3 + 4x^2 - 9)^5}$	8.	$y = x\sqrt{x^3 + 5x + 2}$
9.	$f(x) = x^5 + 3x^3 - (2x + 5)^{10}$	10.	$y = \left(\frac{2x+1}{x-6}\right)^{-5}$
11.	Given the function $f(x) = (3x^2 + 5x - 4)^3$, find $f''(0)$.	12.	Find the equation of the tangent line to the function $y = (x^4 - 15)^5$ at x = 2.

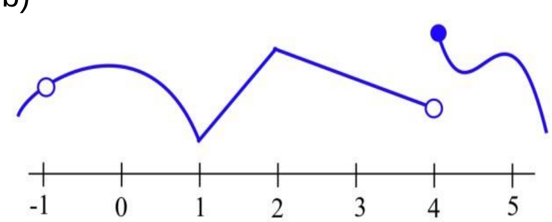
13.	A total revenue function is given to be $R(x) = \sqrt{x^2 - 4x + 4}$ thousands of dollars from the sale of x pianos. a) Find the marginal revenue function. b) Evaluate the marginal revenue function at x = 10 and interpret the answer.
14.	The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C(x) = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes. a) Find the marginal cost $C'(x)$. b) Use Leibniz's notation for the chain rule $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$ to find the rate with respect to time t that the cost is changing. c) Use b) to determine how fast costs are increasing when x = 2 boxes. Include units with the answer.
15.	A company determines that the total cost of producing x units is $C(x) = 10\sqrt{6x^2 + 25} + 200$ hundreds of dollars. a) Find the marginal cost function MC(x). c) Evaluate MC(10) and interpret your answer.

16. Given the graphs below state the points at which the function is not differentiable and why.

a)



b)



Solutions:

1. $f'(x) = 10(2x - 8)^4$

2. $f'(x) = 10(6x - x^2)^9(6 - 2x) = 20(6x - x^2)^9(3 - x)$

3. $f'(x) = (3x + 7)^5 + 15x(3x + 7)^4$

4. $f'(x) = 12(2x + 3)^5 (x - 2)^4 + 4(x - 2)^3 (2x + 3)^6$

5. $f'(x) = \frac{2x+6}{3\sqrt[3]{(x^2+6x-1)^2}}$

6. $\frac{dy}{dx} = \frac{-4x-13}{(x-3)^6}$

7. $f'(x) = \frac{-5(3x^2+8x)}{(x^3+4x^2-9)^6}$

8. $\frac{dy}{dx} = \sqrt{x^3 + 5x + 2} + \frac{x(3x^2+5)}{2\sqrt{x^3+5x+2}}$

9. $f'(x) = 5x^4 + 9x^2 - 20(2x + 5)^9$

10. $\frac{dy}{dx} = -5\left(\frac{2x+1}{x-6}\right)^{-6} \cdot \frac{-13}{(x-6)^2} = \frac{65(x-6)^4}{(2x+1)^6}$

11. -312

12. $y = 160x - 319$

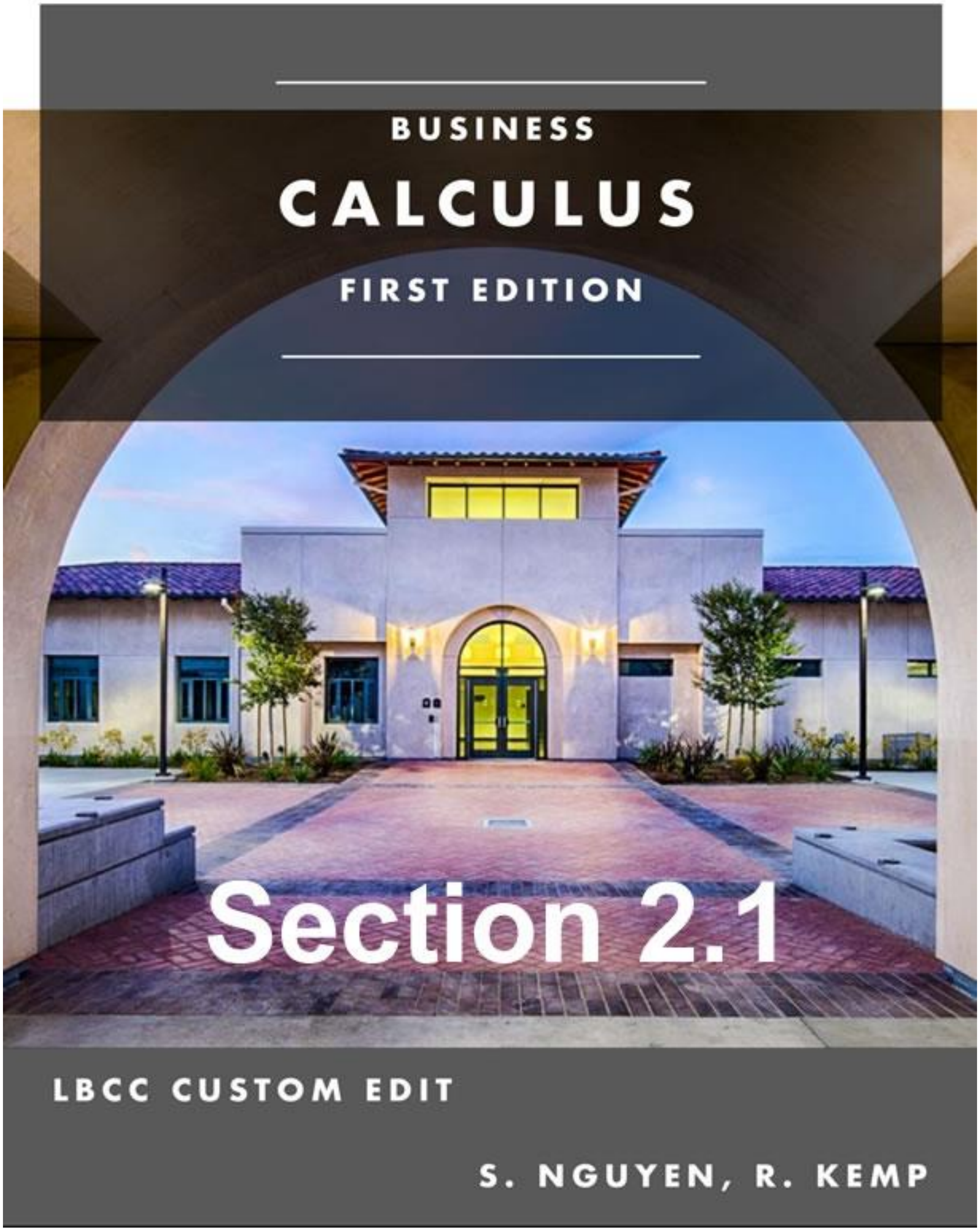
13. a) $MR(x) = \frac{x-2}{\sqrt{x^2-4x+4}}$
   b) $MR(10) = 1$, After 10 pianos are sold the revenue per piano increases at $1000 / piano.

14. a) $C'(x) = 0.0003x^2 - 0.04x + 3 \ \$/box$
   b) $0.03x^2 - 4x + 300 \ \$/week$
   c) $\$ \ 292.12/week$

15. a) $MC(x) = \frac{60x}{\sqrt{6x^2+25}}$
   b) 24. After 10 units are produced the cost increases by $24 per unit.

16. a) At x = -1 because there is a corner, at x = 2 because there is a vertical tangent line, and at x = 1 and x = 4 because the function is not continuous.

   b) At x = 1 and x = 2 because there is a corner, and at x = -1 and x = 4 because the function is not continuous.


BUSINESS
CALCULUS
FIRST EDITION
Section 2.1
LBCC CUSTOM EDIT
S. NGUYEN, R. KEMP

2.1 - FIRST DERIVATIVE. CRITICAL POINTS. RELATIVE EXTREME VALUES

In theory and applications, we often want to maximize or minimize some quantity.

An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance.

A manufacturer may want to maximize profits and market share or minimize waste.

A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Without calculus, we only know how to find the optimum points in a few specific examples (for example, we know how to find the vertex of a parabola).

But what if we need to optimize an unfamiliar function?

The best way we have without calculus is to examine the graph of the function, perhaps using technology.

But our view depends on the viewing window we choose – we might miss something important.

In addition, we'll probably only get an approximation this way. (In some cases, that will be good enough.)

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums, while at the same time ensuring that we haven't missed anything important.

Local/ Relative Maxima and Minima

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.

Definitions:

The function (f) has a **local/relative maximum** at point (a) if:

$f(a) \geq f(x)$ for all (x) near (a)

The function (f) has a **local/relative minimum** at point (a) if:

$f(a) \leq f(x)$ for all (x) near (a)

The function (f) has a **local/relative extreme** at point (a) if:

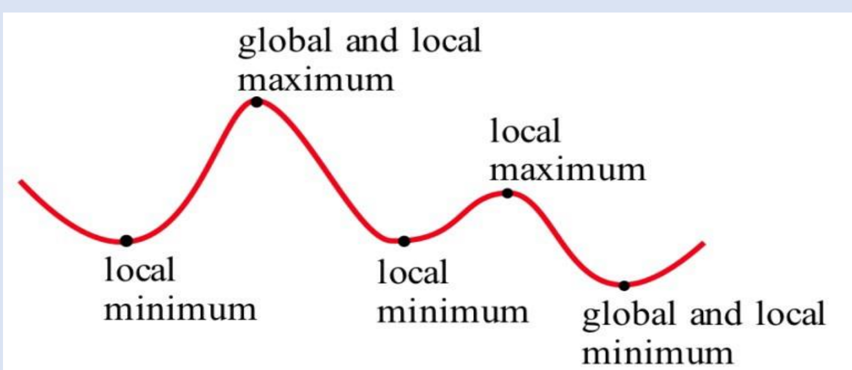
$f(a)$ is a **local/relative maximum or minimum**

The plurals of these are maxima and minima.

We often simply say "max" or "min;" it saves a lot of syllables.

The process of finding maxima or minima is called **optimization.**

A point is a local max or local min, if it is higher (lower) than all the **nearby points**.



Global/ Absolute Maxima and Minima

Next, we will discuss Global/Absolute maximum and minimum values.

Definitions:

The function (f) has a **global/absolute maximum** at point (a) if:

$f(a) \geq f(x)$ for all (x) in the domain of (f)

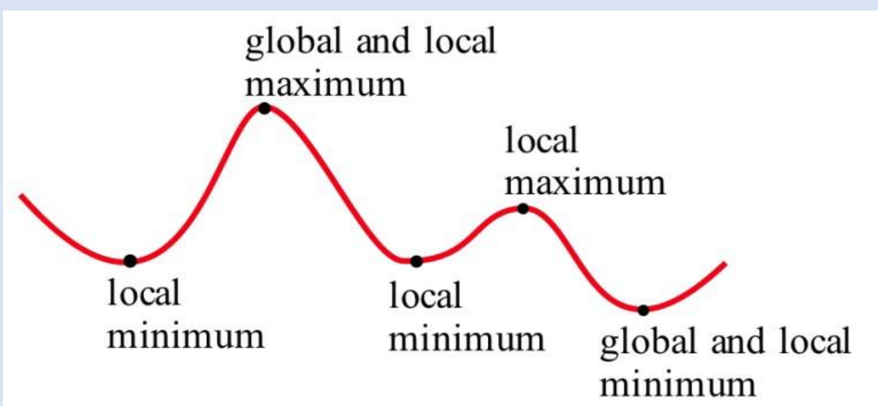
The function (f) has a **global/absolute minimum** at point (a) if:

$f(a) \leq f(x)$ for all (x) in the domain of (f)

The function (f) has a **global/absolute extreme** at point (a) if:

$f(a)$ is a **global/absolute maximum or minimum**

A point is a global/absolute max or global/absolute min if it is higher (lower) than every point on the whole graph.



The local and global extremes of the function below are labeled.

You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes.

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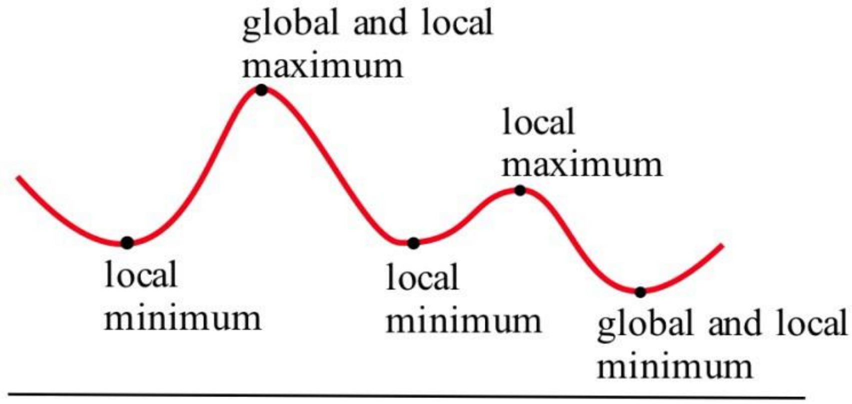


If ($h(x)$) is the height of the earth above sea level at the location (x), then the global maximum of (h), is:

h(summit of Mt. Everest) = 29,028 feet.

A local maximum of (h) for the United States is:

h(summit of Mt. McKinley) = 20,320 feet.

A local minimum of h for the United States is:

h(Death Valley) = – 282 feet.

Example Problem #1

The table shows the annual calculus enrollments at a large university.

Which years had local maximum or minimum calculus enrollments?

What were the global maximum and minimum enrollments in calculus?

year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
enrollment	1257	1324	1378	1336	1389	1450	1523	1582	1567	1545	1571

Solution:

There were local maxima in 2002 and 2007; the global maximum was 1582 students in 2007.

There were local minima in 2003 and 2009; the global minimum was 1324 students in 2001.

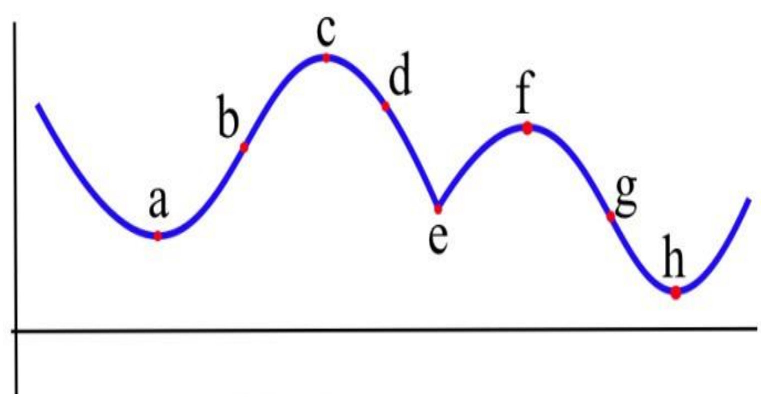
We cannot really think of 2000 as a local minimum or 2010 as a local maximum, since we have no data before 2000 or after 2010, however though, some books would include the endpoints.

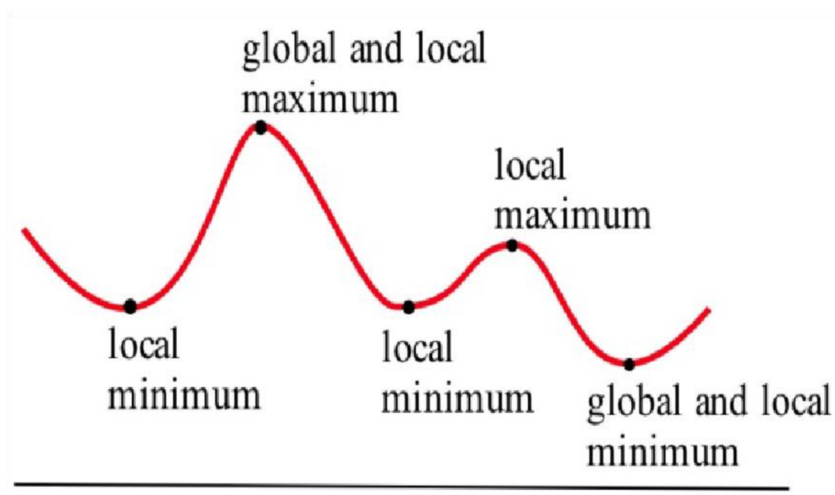
Finding Maxima and Minima of a Function

What must the tangent line look like at a local or relative max or min?

Look at these two graphs again – you'll see that at the extreme points (a, c, f, and h), the tangent line is horizontal (so f' = 0).

There is one corner in the blue graph at point e – the tangent line cannot be sketched there, so (f') is undefined. That gives us the clue how to find extreme values.





Definition

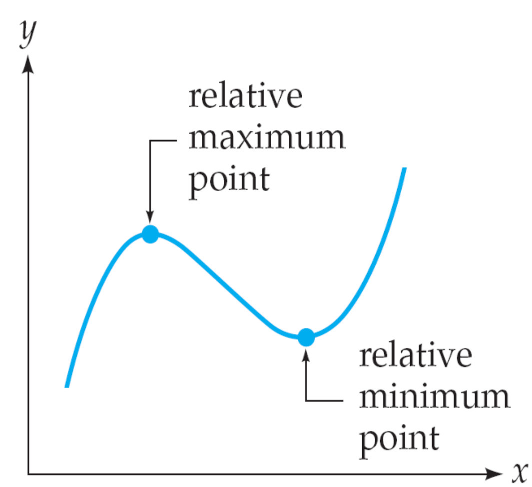
A **Critical Number** for a function (f), is a value $(x = a)$ in the domain of (f) where either:

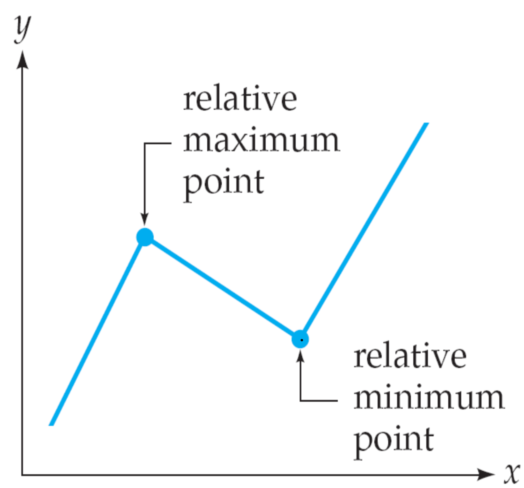
$$f'(a) = \frac{d}{dx}f(a) = 0 \quad \text{or} \quad f'(a) = \frac{d}{dx}f(a) = Undefined$$

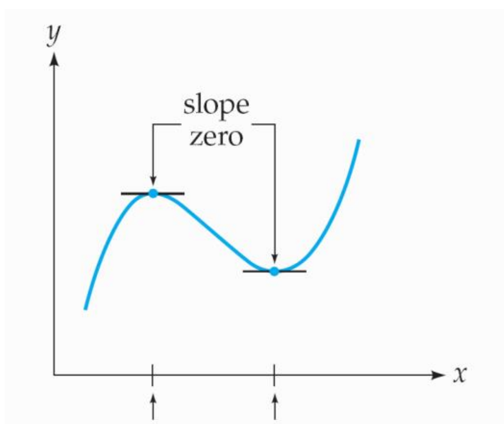
A **Critical Point** for a function (f), is a point $(a, f(a))$,

where (a) is, a **critical number** of (f).

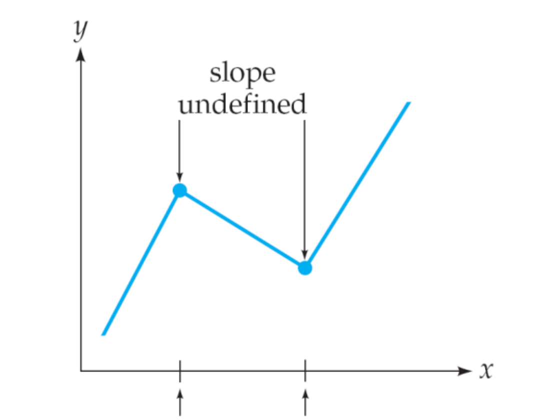
Useful Fact: A local/relative max or min of (f) can only occur at a critical point.







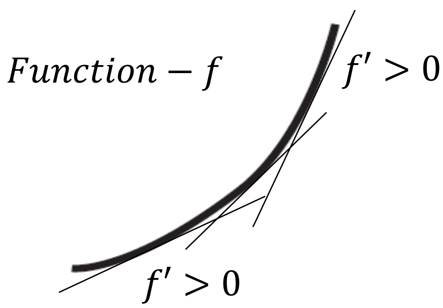
This function has two critical numbers (where the derivative is zero).

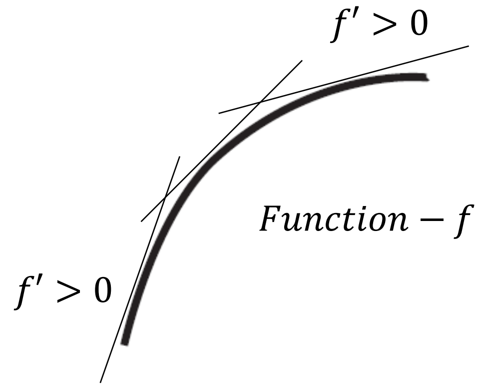


This function also has two critical numbers (but where the derivative is undefined).

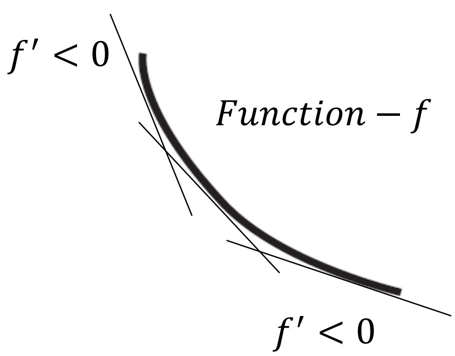
Below we sketch the curve of function (f), by estimating the first derivatives as increasing or decreasing "slopes" (f') on the curves.

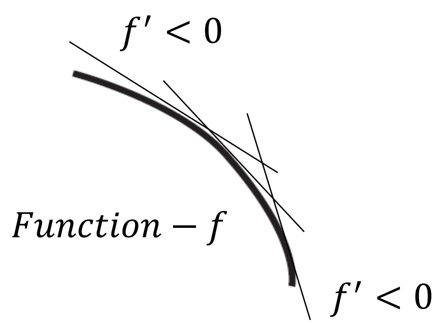
1. ***The slope/derivative is Increasing*** *($f' > 0$)*





2. ***The slope/derivative is Decreasing*** *($f' < 0$)*



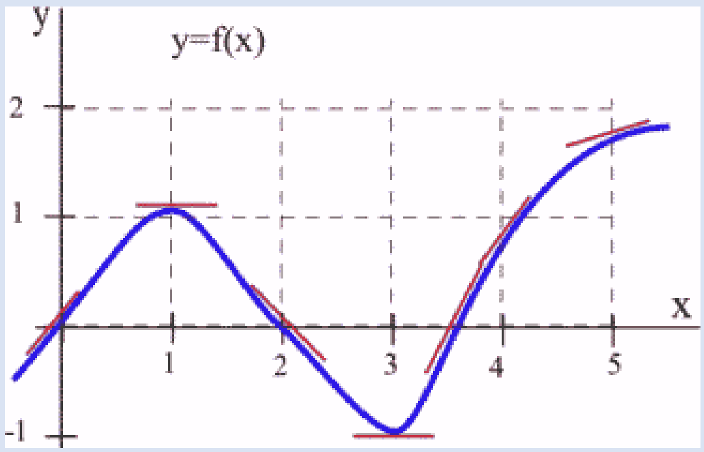


Example Problem #2

Given

Below is the graph of a function ($y = f(x)$).

We can use the information in the graph to fill in a table showing values of ($f'(x)$):



At various values of x, draw your best guess at the tangent line and measure its slope. You might have to extend your lines so you can read some points.

In general, your estimate of the slope will be better if you choose points that are easy to read and far away from each other.

Here are estimates for a few values of x (parts of the tangent lines used are shown above in the graph):

x	$y = f(x)$	$f'(x)$ = the estimated *slope* of the tangent line to the curve at the point (x, y).
0	0	1
1	1	0
2	0	-1
3	-1	0
3.5	0	1

Example Problem #2 – Cont'd

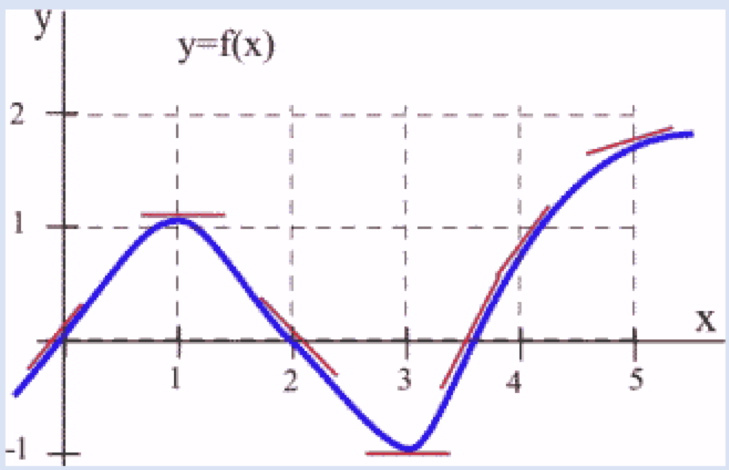
We can estimate the values of ($f'(x)$) at some non-integer values of (x), too:

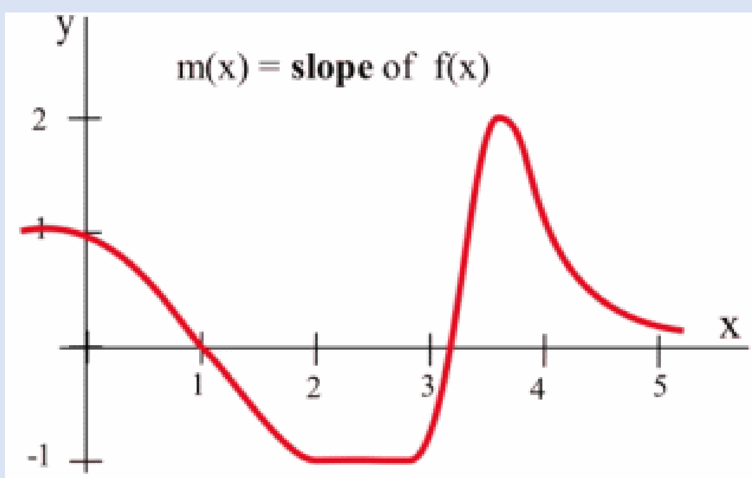
$$f'(0.5) \approx 0.5 \quad \text{and} \quad f'(1.3) \approx -0.3$$

We can even think about entire intervals.

For example, if ($0 < x < 1$), then ($f(x)$) is increasing, all the slopes are positive, and so ($f'(x)$) is positive.

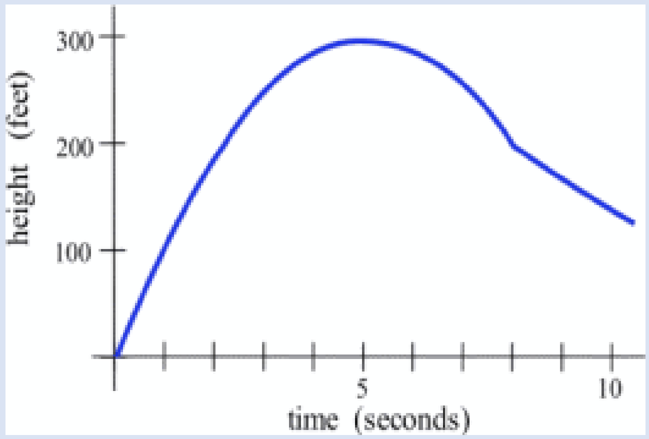
The values of ($f'(x)$) depend on the values of (x), and ($f'(x)$) is a function of (x). We can use the results in the table to help sketch the graph of ($f'(x)$).





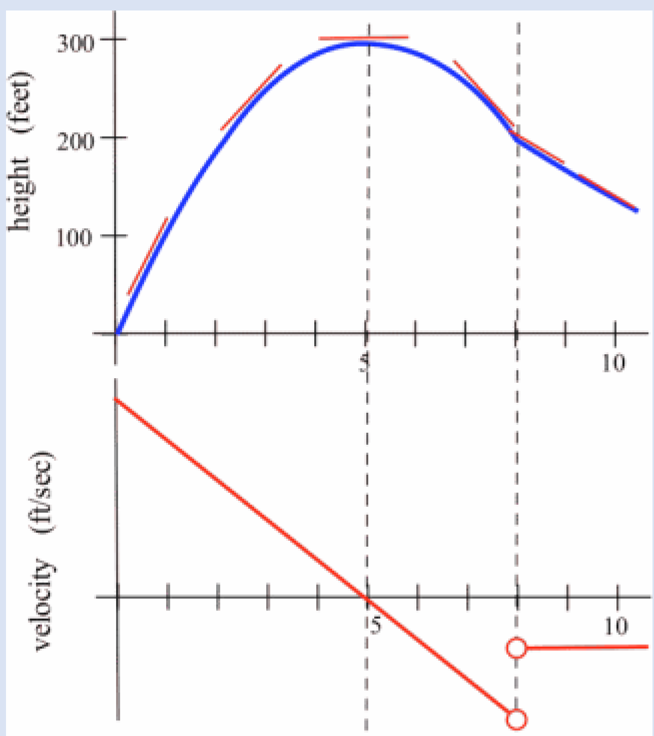
Example Problem #3

Shown is the graph of the height ($h(t)$) of a rocket at time (t).



Sketch the graph of the velocity of the rocket at time (t). (Velocity is the derivative of the height function, so it is the slope of the tangent to the graph of position or height.)

We can estimate the slope of the function at several points. The lower graph below shows the velocity of the rocket. This is $h'(t) = v(t)$.



Example Problem #4

Find the Critical Numbers given the following **derivative** functions ($f'(x)$):

a) The derivative function ($f'(x)$):

$$f'(x) = 3 \cdot (x + 2)(x - 4) = 0$$

Solution:

The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = -2 & \rightarrow f' = 0 \\ x = 4 & \rightarrow f' = 0 \end{cases}$$

b) The derivative function ($f'(x)$):

$$f'(x) = 2x \cdot \left(x - \frac{2}{5}\right)(2x + 7) = 0$$

Solution:

The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = 0 & \rightarrow f' = 0 \\ x = \frac{2}{5} & \rightarrow f' = 0 \\ x = -\frac{7}{2} & \rightarrow f' = 0 \end{cases}$$

c) The derivative function ($f'(x)$):

$$f'(x) = 4x^2 \cdot (x^2 - 4)(2x - 1) = 0$$

Solution:

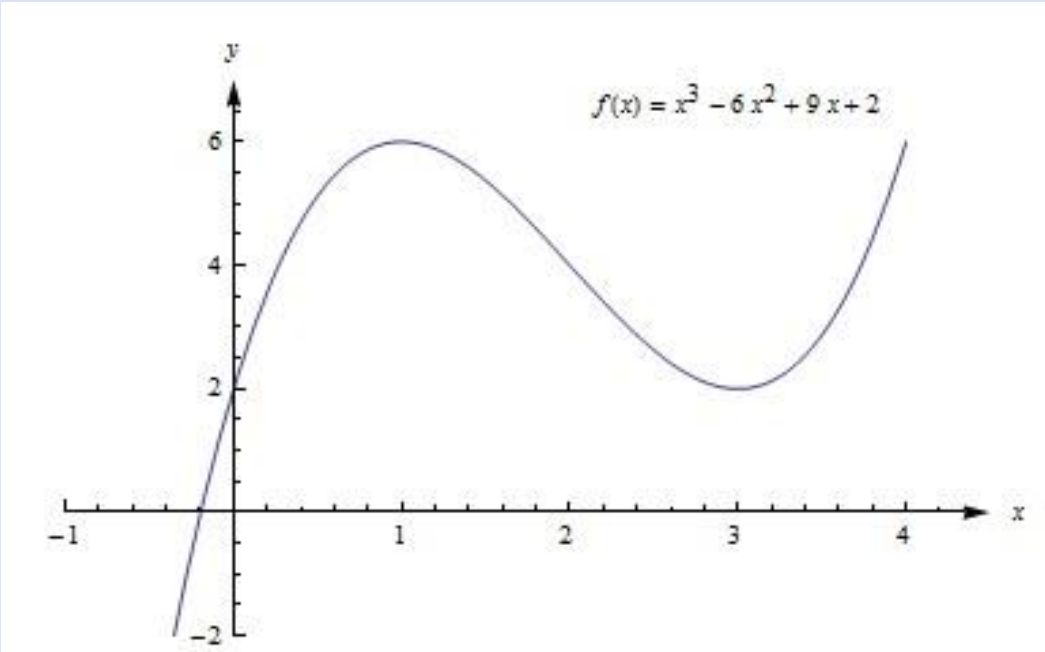
The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = 0 & \rightarrow f' = 0 \\ x = 2 & \rightarrow f' = 0 \\ x = -2 & \rightarrow f' = 0 \\ x = \frac{1}{2} & \rightarrow f' = 0 \end{cases}$$

Example Problem #5

Find the critical points for the function:

$$f(x) = x^3 - 6x^2 + 9x + 2$$



Solution:

A critical number of ($f(x)$) can occur only where:

$$f'(x) = 0 \quad \text{or where} \quad f'(x) = Undefined.$$

$$f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 9x + 2) = 3x^2 - 12x + 9$$

The Critical Numbers (CN) for ($f(x)$) occur only where ($f'(x) = 0$):

$$f'(x) = 3(x^2 - 4x + 3) = 3(x-1)(x-3)) = 0$$

$$CN = \begin{cases} x = 1 \\ x = 3 \end{cases}$$

There are no places where ($f'(x) = Undefined$).

Example Problem #5 – Cont'd

The critical numbers are ($x = 1$) and ($x = 3$).

$$CN = \begin{cases} x = 1 \\ x = 3 \end{cases}$$

To find the critical points we need to find the y-values, by substituting ($f(1)$) and ($f(3)$).

$$y = f(x) = x^3 - 6x^2 + 9x + 2$$

Substituting ($f(1)$);

$$y = f(1) = (1)^3 - 6(1)^2 + 9(1) + 2 = 1 - 6 + 9 + 2 = 6$$

Substituting ($f(3)$);

$$y = f(3) = (3)^3 - 6(3)^2 + 9(3) + 2 = 9 - 54 + 27 + 2 = 2$$

So, the critical points are (1, 6) and (3, 2).

$$CP = \begin{cases} \big(1,\ f(1)\big) = (1, 6) \rightarrow \mathit{Local\ Maximum} \\ \big(3,\ f(3)\big) = (3, 2) \rightarrow \mathit{Local\ Minimum} \end{cases}$$

These are the only possible locations of local extremes of ($f(x)$).

We haven't discussed yet how to tell whether either of these points is a local extreme of ($f(x)$), or which kind it might be.

But we can be certain that no other point is a local extreme.

The graph of ($f(x)$) shows that at the point ($\big(1,\ f(1)\big) = (1, 6)$) is a **local maximum**. And ($\big(3,\ f(3)\big) = (3, 2)$) is a **local minimum**.

This function does not have a global maximum or minimum.

Remember this example! It is not enough to find the critical points -- we can only say that (f) **might have** a local extreme at the critical points.

Is that critical point a Maximum or Minimum (or Neither)?

Example Problem #6

Find the local extreme points of:

$$f(x) = x^3$$

Solution:

To find the local extreme point we need to find where.

$$f'(x) = 0 \quad \text{or where} \quad f'(x) = Undefined.$$

$$f'(x) = 3x^2$$

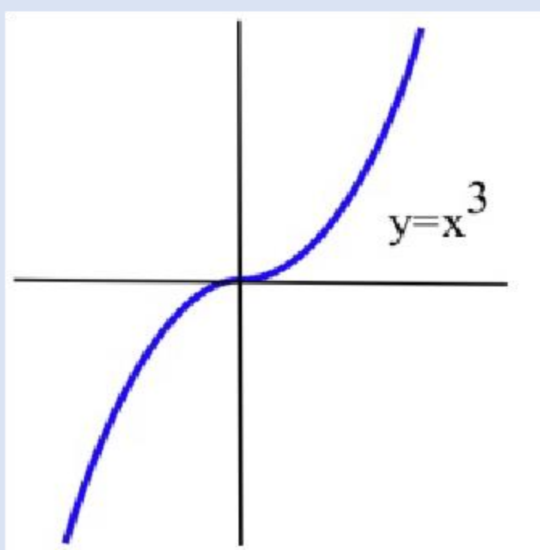
The Critical Numbers (CN) for $(f(x))$ occur only where $(f'(x) = 0)$:

$$f'(x) = 3x^2 = 0$$

So, this is our critical number, and the critical point is (0, 0).

$$CN = \{x = 0$$

We can best see what is happening by looking at the graph of $(f(x) = x^3)$.



Example Problem #6 – Cont'd

So, the critical points are.

$$CP = \{(0,\ f(0)) = (0,0) \rightarrow Neither\ Local\ Max/Min$$

Let's look at what happens to the graph right of ($CP(0,0)$).

$$If \quad x > 0, \quad then \quad f\,(x) = x^3 > 0$$

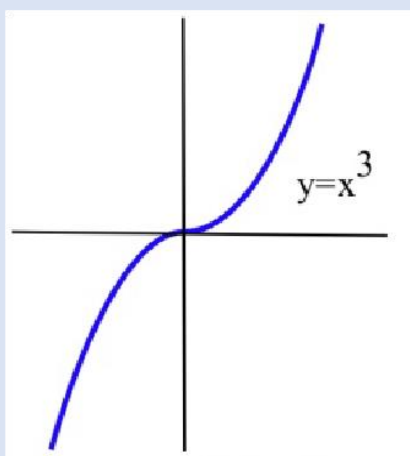
$$f'(x) = 3x^2 > 0 \quad \rightarrow \quad Increasing$$

Let's look at what happens to the graph left of ($CP(0,0)$).

$$If \quad x < 0, \quad then \quad f\,(x) = x^3 < 0$$

$$f'(x) = 3x^2 > 0 \quad \rightarrow \quad Increasing$$

So, as we see the y - value of the critical point, which is 0 is not greater than or smaller than all the y - values around the critical point, which means that (0, 0) is neither a local minimum nor a local maximum.



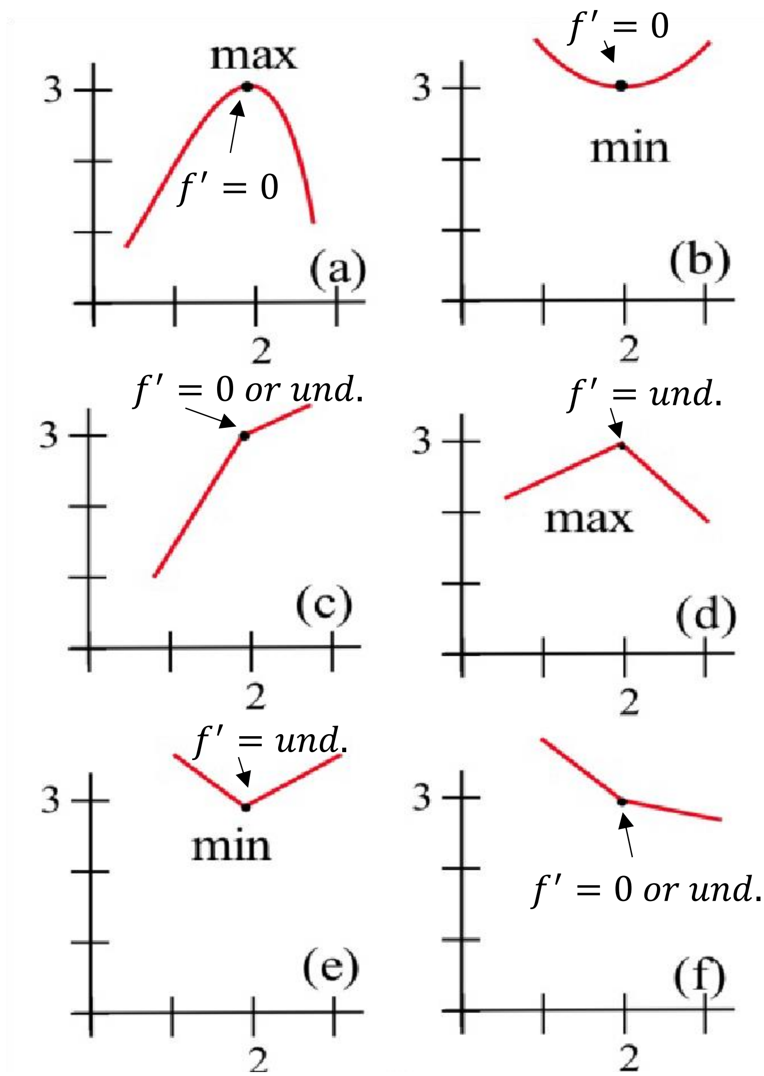
Once we have found the critical points of ($f\,(x)$), we still have the problem of determining whether these points are maxima, minima or neither.

All, of the graphs below have a critical point at (2, 3).

It is clear from the graphs that the point (2,3) is a **local maximum** in (a) and (d).

The point (2,3) is a **local minimum** in (b) and (e).

And, the point and (2,3) **is not a local extreme** in (c) and (f).



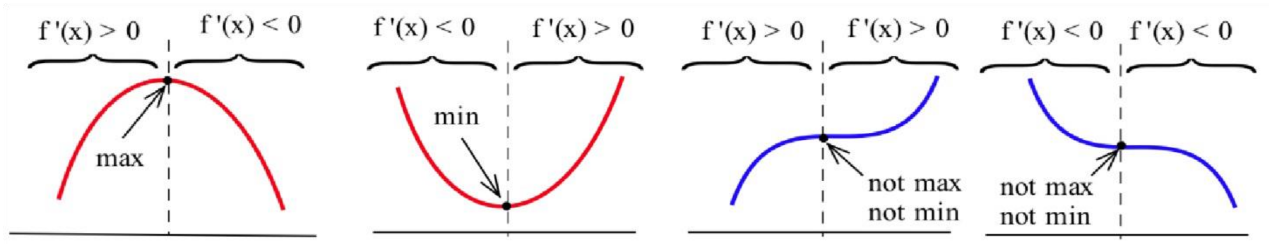
The critical numbers only give the **possible** locations of extremes, and some critical numbers are not the locations of extremes.

The critical numbers are the **candidates** for the locations of maxima and minima.

f ' and Extreme Values of f

Four possible shapes of graphs are shown here – in each graph, the point marked by an arrow is a critical point, where ($f'(x) = 0$).

What happens to the derivative near the critical point?



At a local max, such as in the graph on the left, the function increases on the left of the local max, then decreases on the right.

The derivative is first positive, then negative at a local max.

At a local min, the function decreases to the left and increases to the right, so the derivative is first negative, then positive.

When there isn't a local extreme, the function continues to increase (or decrease) right past the critical point – the derivative doesn't change its sign.

The First Derivative Test for Extremes:

Find the critical points of ($f(x)$).

For each **Critical Number** (CN), examine the sign of (f') to the left and to the right of (CN).

What happens to the sign as you move from left to right?

If ($f'(x)$) changes from positive to negative at ($x = CN$), then ($f(x)$) has a **local maximum** at ($CN, f(CN)$).

If ($f'(x)$) changes from negative to positive at ($x = CN$), then ($f(x)$) has a **local minimum** at ($CN, f(CN)$).

If ($f'(x)$) does not change sign at ($x = CN$), then ($CN, f(CN)$) is **neither a local max nor a local min**.

Example Problem #7

Find the critical points of ($f(x)$) and classify them as local max, local min, or neither.

$$f(x) = x^3 - 6x^2 + 9x + 2$$

Solution:

We already found the critical points; they are (1, 6) and (3, 2).

Now we can use the first derivative test to classify each. Recall that

$$f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 9x + 2) = 3x^2 - 12x + 9$$

The Critical Numbers (CN) for ($f(x)$) occur only where ($f'(x) = 0$):

$$f'(x) = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)) = 0$$

$$CN = \begin{cases} x = 1 \\ x = 3 \end{cases}$$

The factored form is easiest to work with here, so let's use that; and choose Test Points around the Critical Numbers (CN).

x	$f'(x) = 3(x-1)(x-3))$	$f'(x) \rightarrow$ Increase/Decrease
0	$f'(0) = 3(0-1)(0-3)) = +9$	$f'(0) > 0 \rightarrow Increasing$
1	$f'(1) = 0$	$f'(1) = 0 \rightarrow Local\ Max$
2	$f'(2) = 3(2-1)(2-3)) = -3$	$f'(2) < 0 \rightarrow Decreasing$
3	$f'(3) = 0$	$f'(3) = 0 \rightarrow Local\ Min$
4	$f'(4) = 3(4-1)(4-3)) = +9$	$f'(4) > 0 \rightarrow Increasing$

Example Problem #7 – Cont'd

You choose a number slightly less than $(CN = 1)$, use $(x = 0)$, or $(x = 0.9)$; and slightly greater than $(CN = 1)$, use $(x = 2)$, or $(x = 1.5)$ to plug into the formula for $(f'(x))$.

Next, you choose a number slightly less than $(CN = 3)$, use $(x = 2)$, or $(x = 1.5)$; and slightly greater than $(CN = 3)$, use $(x = 4)$, or $(x = 3.5)$ to plug into the formula for $(f'(x))$.

Then you examine its sign. But we don't care about the numerical value, all we are interested in is its sign.

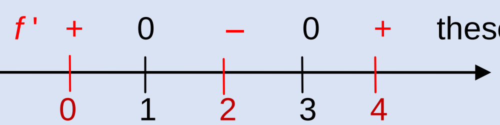
Thus, $(f'(x))$ changes from positive to negative, around the $(CN = 1)$ so there is a local max at (1, 6).

Thus, $(f'(x))$ changes from negative to positive, around the $(CN = 3)$ so there is a local min at (3, 2).

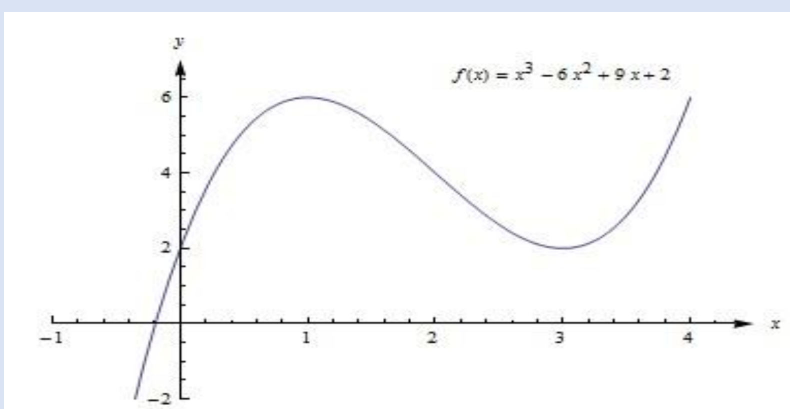
As another approach, we could draw a number line, and mark the critical numbers.

We already know the derivative is zero or undefined at the critical numbers.

On each interval between these values,



the derivative will stay the same sign. This confirms what we saw before in the graph.



Curve Sketching

This section examines some of the interplay between the shape of the graph of (f), and the behavior of (f').

First Derivative Information about the Values

Definitions:

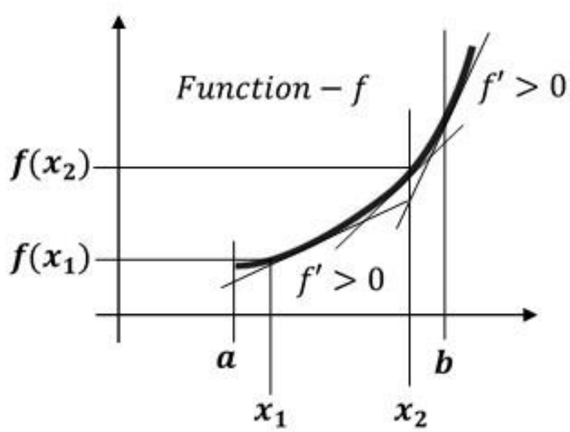
The function (f) is **Increasing** on interval $[a, b]$ if:

$$a < x_1 < x_2 < b \quad implies \quad f(x_1) < f(x_2)$$

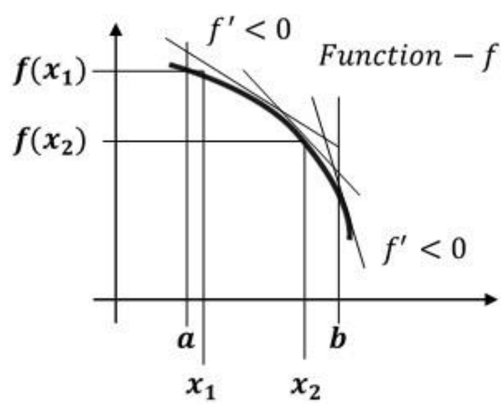
The function (f) is **Decreasing** on interval $[a, b]$ if:

$$a < x_2 < x_1 < b \quad implies \quad f(x_1) > f(x_2)$$

The function (f) is **Increasing** on interval $[a, b]$



The function (f) is **Decreasing** on interval $[a, b]$



If we have a graph of, (f), we will see what we can **conclude about the values** of (f').

First Derivative Information about Shape

For a function (f) which is differentiable on an interval $[a, b]$;

(i) If function (f) is **Increasing** on interval $[a, b]$, then

$$f'(x) \geq 0 \quad for\ all\ (x)\ \ in\ the\ inerval\ [a, b]$$

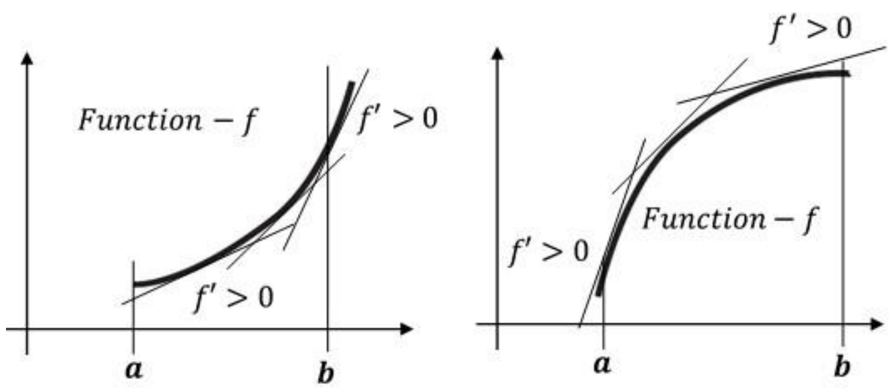
(ii) If function (f) is **Decreasing** on interval $[a, b]$, then

$$f'(x) \leq 0 \quad for\ all\ (x)\ \ in\ the\ inerval\ [a, b]$$

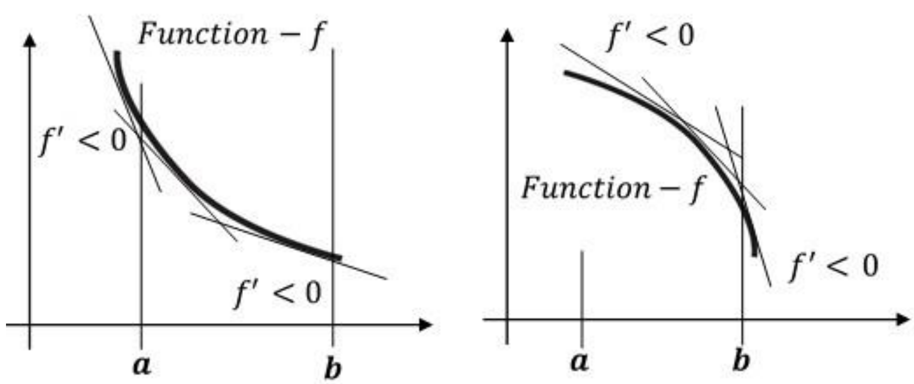
(iii) If function (f) is **Constant** on interval $[a, b]$, then

$$f'(x) = 0 \quad for\ all\ (x)\ \ in\ the\ inerval\ [a, b]$$

The slope/derivative is positive, and the function is Increasing $(f' > 0)$



The slope/derivative is negative, and the function is Decreasing $(f' < 0)$



If we know values of (f'), we will see what we can **conclude about the graph** of (f).

The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective.

It says that if we know something about the values of f ' then we can draw some conclusions about the graph of (f).

First Derivative Information about Shape (Part 2)

For a function (f) which is differentiable on an interval [$I = [a, b]$];

i) $If\ f(x) > 0 \quad for\ all\ (x)\ \ in\ the\ interval, I = [a, b], then$

$$f(x)\ is\ increasing\ on\ I$$

$ii)\ If\ f'(x) < 0 \quad for\ all\ (x)\ \ in\ the\ interval, I = [a, b], then$

$$f(x)\ is\ Decreasing\ on\ I$$

$iii)\ If\ f'(x) = 0 \quad for\ all\ (x)\ \ in\ the\ inerval, I = [a, b], then$

$$f(x)\ is\ Constant\ on\ I$$

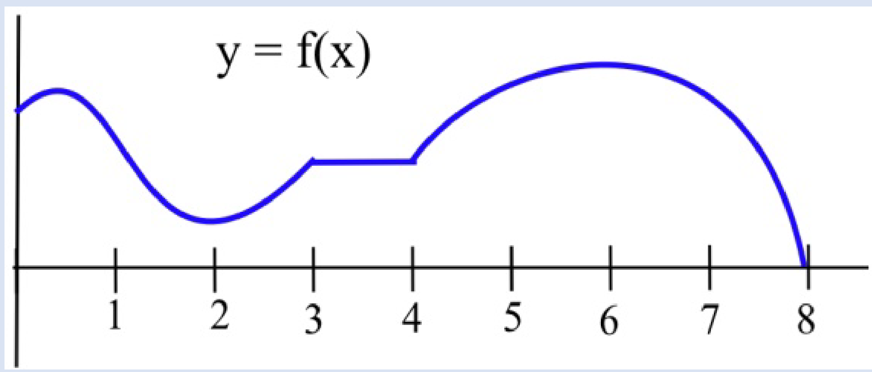
Graphically, (f) is **increasing** (decreasing) if, as we move from left to right along the graph of (f), the height of the graph **increases** (decreases).

These same ideas make sense if we consider (t) to be the height (in feet) of a rocket at time (t) seconds.

We naturally say that the rocket is rising or that its height is increasing if the height $(h(t))$ increases over a period of time, as (t) increases.

Example Problem #8

List the intervals on which the function shown increasing or decreasing.



Solution:

The function (f) is increasing on the intervals [0,0.5], [2,3] and [4,6].

The function (f) is decreasing on [0.5,2] and [6,8].

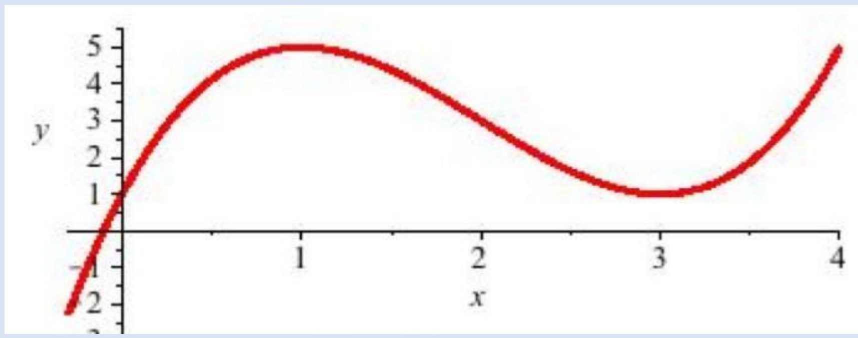
On the interval [3,4] the function is not increasing or decreasing, it is constant.

Example Problem #9

Use information about the values of ($f'(x)$) to help graph:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Solution:



The derivative ($f'(x)$) is a polynomial so it is always defined.

$$f'(x) = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)) = 0$$

So ($f'(x) = 0$) only when ($x = 1$) or ($x = 3$).

Then substituting the critical numbers into the original function yiels the critical points which are (1, 5) and (3, 1).

On each of these intervals, the function is either always increasing or always decreasing.

If (x < 1), then ($f'(x) \geq 0$) so ($f(x)$) is **increasing**.

If (1 < x < 3), then ($f'(x) \leq 0$) so ($f(x)$) is **decreasing**.

If (x > 3), then ($f'(x) \geq 0$) so $f(x)$ is **increasing**.

The only critical numbers for ($f(x)$) are when ($x = 1$) and ($x = 3$), and they divide the real number line into three intervals: ($-\infty$, 1), (1, 3), and (3, ∞).

Example Problem #9 – Cont'd

Even though we don't know the value of $(f(x))$ anywhere yet we do know a lot about the shape of the graph of $(f(x))$.

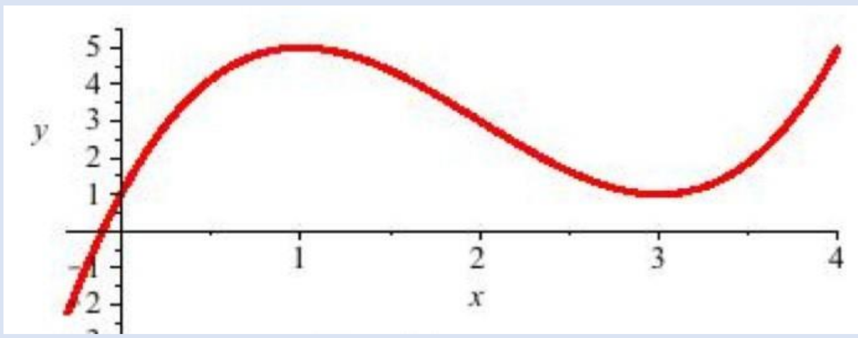
As we move from left to right along the (x–axis), the graph of $(f(x))$ increases until (x = 1), then the graph decreases until (x = 3), and then the graph increases again.

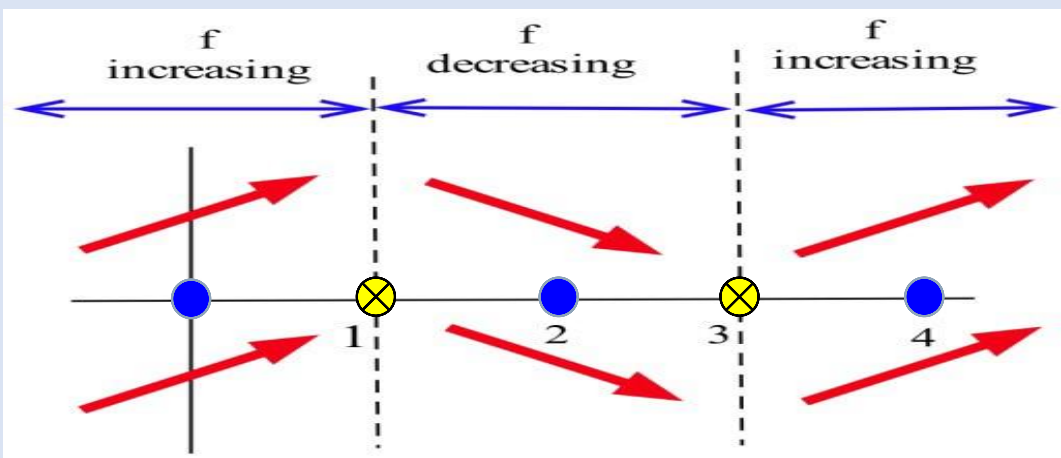
Find Open Intervals of Increase (OI) & Decrease (OD), and finalize results:

$$\text{Open Interval of Increase} \rightarrow OI \quad = \quad (-\infty, 1), (3, \infty)$$

$$\text{Open Interval of Decrease} \rightarrow OD \quad = \quad (1, 3)$$

The resulting graph of $(f(x))$ is shown here.





Example Problem #9 – Cont'd

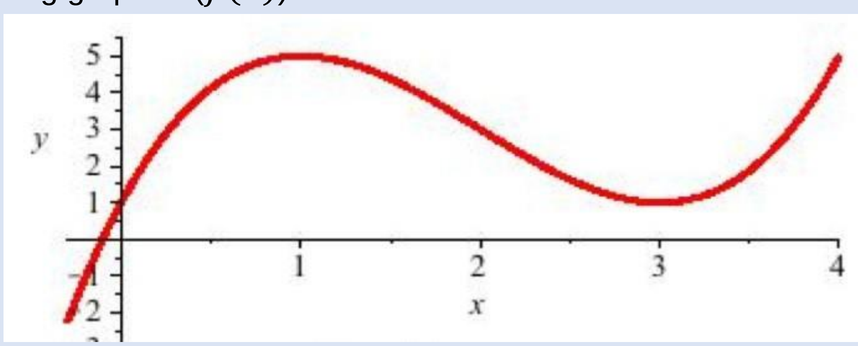
The graph of $(f(x))$ makes "turns" when (x = 1) and (x = 3); it has a local maximum at (x = 1), and a local minimum at (x = 3). See Graph & Table:

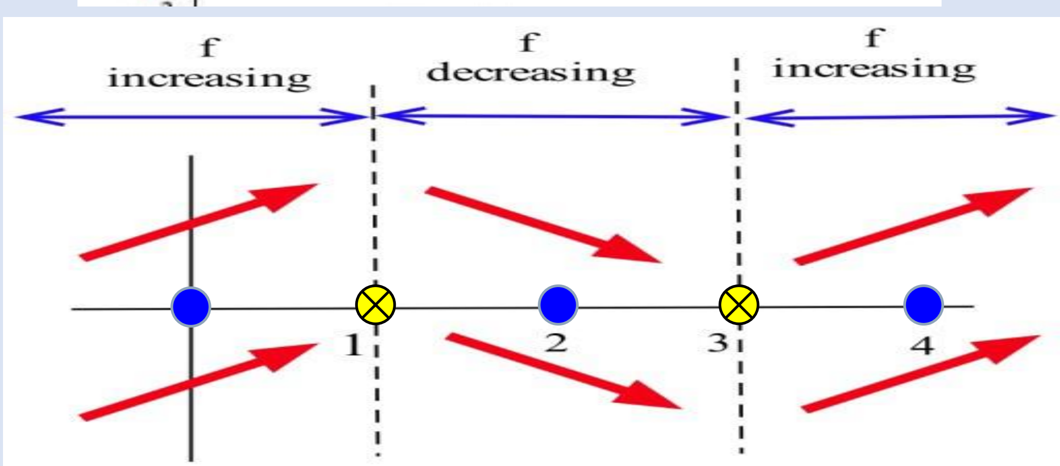
x	$f'(x) = 3(x-1)(x-3))$	$f'(x) \rightarrow$ Increase/Decrease
0	$f'(0) = 3(0-1)(0-3)) = +9$	$f'(0) > 0 \rightarrow Increasing$
1	$f'(1) = 0$	$f'(1) = 0 \rightarrow Local\ Max$
2	$f'(2) = 3(2-1)(2-3)) = -3$	$f'(2) < 0 \rightarrow Decreasing$
3	$f'(3) = 0$	$f'(3) = 0 \rightarrow Local\ Min$
4	$f'(4) = 3(4-1)(4-3)) = +9$	$f'(4) > 0 \rightarrow Increasing$

To plot the graph of $(f(x))$, we still need to evaluate $(f(x))$ at a few values of (x), but only at a very few values.

$f(1) = 5$, and (1,5) is a local maximum of $(f(x))$.
$f(3) = 1$, and (3,1) is a local minimum of $(f(x))$.

The resulting graph of $(f(x))$ is shown here.





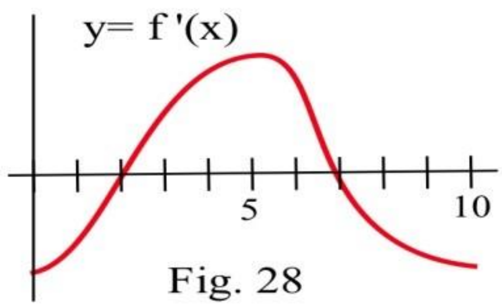
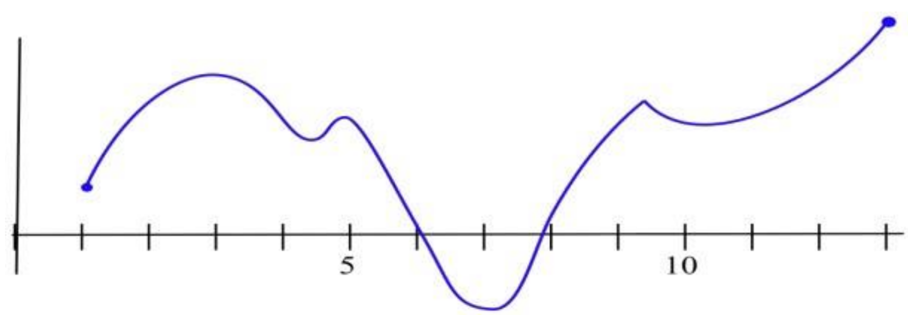
2.1 - EXERCISES

Find the critical numbers and then the critical points of the functions.

1.	$y = 3x^2 - 2x^3$	2.	$f(x) = x^3 - 3x^2 + 6$
3.	$f(x) = \frac{x^3}{3} - 5x^2 + 24x$	4.	$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$
5.	$f(x) = (x+2)^{2/3} + 1$	6.	$f(x) = \sqrt[3]{x+1}$

Sketch the graph of the following functions, by finding the local mins and maximums and the intervals of increase and decrease and making a sign diagram for the first derivative.

7.	$y = 3x^2 - 2x^3$	8.	$f(x) = x^2 + 8x + 7$
9.	$y = x^3 + 3x^2 - 9x - 3$	10.	$f(x) = \frac{1}{3}x^3 - 3x^2 + 9x - 1$

11.	A town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function: $$P(t) = -t^3 + 27\,t + 30$$ where t is years since 2000. Find the relative extreme values of this function and sketch the graph for $0 \leq t \leq 5$
12.	The cost of producing an x = amount of desks in hundreds, can be modeled by the function $$C(x) = x^4 - 8x^2 + 20,$$ where C is in thousands of dollars. Find the number of desks produced that will minimize the cost (i.e, the relative minimum) and graph the function on the interval $0 \leq x \leq 5$. (Remember that you cannot produce a negative amount of desks.)
13.	Given below is the graph of the **derivative** of a continuous function f. (a) List the critical numbers of f. (b) For what values of x does f have a local maximum? (c) For what values of x does f have a local minimum?  Fig. 28
14.	Find all of the critical numbers of the function shown and identify them as local max, local min, or neither. Find the global max and min on the interval. 

Solutions:

1. Critical numbers: x = 0, x = 1, Critical points (0, 0), (1, 1)

2. Critical numbers: x = 0, x = 2, Critical points (0, 6), (2, 2)

3. Critical numbers: x = 4, x = 6, Critical points (4, $\frac{112}{3}$), (6, 36)

4. Critical numbers: x = 0, x = 1, x = -2, Critical points (0, 0), (1, $-\frac{5}{12}$), (-2, $-\frac{8}{3}$)

5. Critical numbers: x = -2, Critical points (-2, 1)

6. Critical numbers: x = -1, Critical points (-1, 0)

For graphs please see desmos.com or a graphing calculator.

7. Local min (0, 0), Local max (1, 1), increase on (0, 1), decrease on $(-\infty, 0)$, $(1, \infty)$

8. Local min (-4, -9), increase on $(-4, \infty)$, decrease on $(-\infty, -4)$

9. Local min (1, -8), Local max (-3, 24), increase on $(-\infty, -3)$, $(1, \infty)$, decrease (-3, 1)

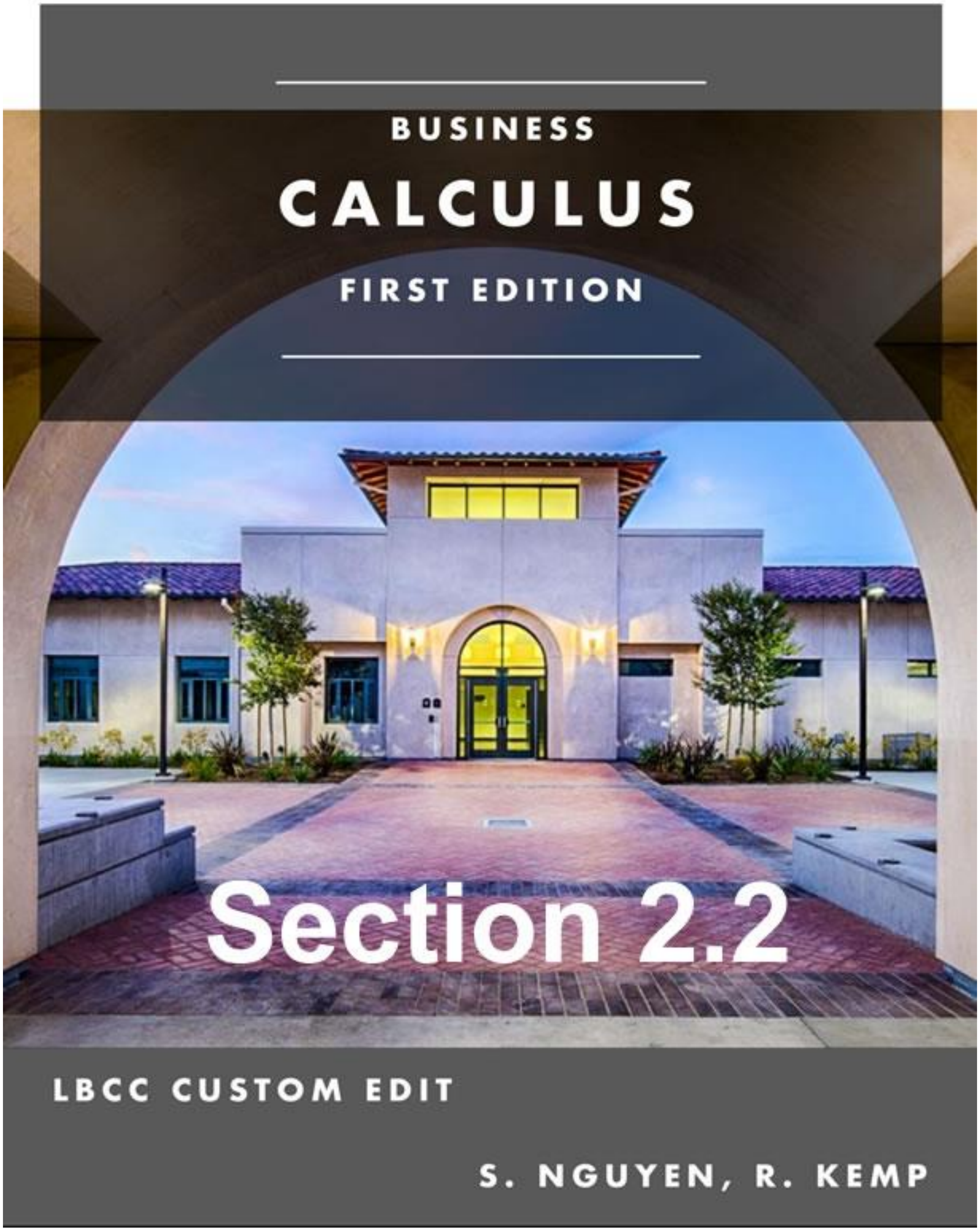
10. No local min or local max, increase on $(-\infty, \infty)$

11. Local max (3, 84)

12. Local minimum at (2, 4). The company must produce 200 desks to minimize the cost, which will be $4,000.

13. a) x = 2, x = 7, b) x =7, c) x = 2

14. At x = 3, 5, and 9.5 we have local maximums, at x = 4.5, 7 and 10.5 we have local minimums.


BUSINESS
CALCULUS
FIRST EDITION
Section 2.2
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S. NGUYEN, R. KEMP

2.2 - FIRST DERIVATIVE. CRITICAL POINTS. RELATIVE EXTREME VALUES – RATIONAL FUNCTIONS

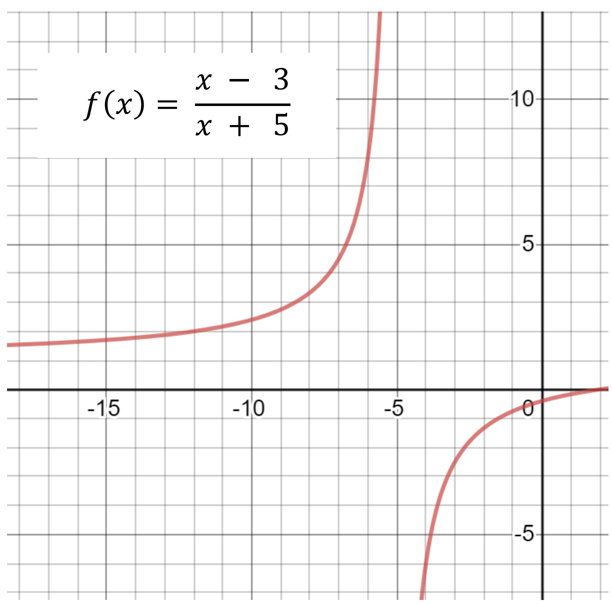
Rational Functions and Curve Sketching

In this section we will graph Rational Functions. Rational Functions unlike polynomial functions are the ratio of two independent functions.

The rational function is a two-fold function in that it has an independent numerator function $h(x)$ and an independent denominator function $g(x)$.

$$y = f(x) = \frac{h(x)}{g(x)} = \frac{x - 3}{x + 5}$$

When we analyze rational functions, we find or identify the function's Vertical Asymptote and its Horizontal Asymptote.



If we have a graph of the Rational Function, (f), we will see what we can **conclude about the values** of (f'). Likewise, if we know values of (f'), we will see what we can **conclude about the graph** of (f).

Definitions:

A rational function is a function of the form

$$y = f(x) = \frac{h(x)}{g(x)}$$

where both h(x) and g(x) are independent functions. Graphs of rational functions have vertical and sometimes horizontal asymptotes.

Test for Vertical Asymptote, at $(x = a)$, such that $[g(a) = 0 \; and \; h(a) \neq 0]$

$$y = f(a) = \frac{h(a)}{0} = Undefined$$

$$\boldsymbol{Vertical\ Asymptote} \rightarrow VA: \quad x = a$$

The Rational Function (f) has a Vertical Asymptote at any value $(x = a)$, when $[g(a) = 0]$ and $[h(a) \neq 0]$.

If both $g(a) = 0$ and $h(a) = 0$, then the function has a hole in the graph at $x = a$**.**

The **Horizontal Asymptote** exits when the **Rational Function** (f) is:

$$y = \lim_{x \to \infty} f(x) = \frac{h(\infty)}{g(\infty)} = L$$

and, or

$$y = \lim_{x \to -\infty} f(x) = \frac{h(-\infty)}{g(-\infty)} = L$$

$$\boldsymbol{Horizontal\ Asymptote} \rightarrow HA: \quad y = L$$

The Rational Function (f) has a Horizontal Asymptote if the limit as (x) approaches infinity is a value ($y = L$).

Since we have not discussed much finding limits at infinity, such as $\lim_{x \to \infty} f(x)$, recall from your College Algebra classes a different way of determining Horizontal asymptotes.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at $y = 0$

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Degree of denominator < degree of numerator: No horizontal asymptote

Critical Numbers – Rational Functions

Example Problem #1

Find the Critical Numbers given the following derivative functions ($f'(x)$):

d) The derivative function ($f'(x)$):

$$f'(x) = \frac{9 \cdot (x - 12)}{(x + 7)} = 0$$

Solution:

The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = 12 & \rightarrow f' = 0 \\ x = -7 & \rightarrow f' = \dfrac{-171}{0} = Undefined \end{cases}$$

Find the Critical Values for the following derivative functions ($f'(x)$):

e) The derivative function ($f'(x)$):

$$f'(x) = \frac{45 \cdot x^2}{(x - 1)(x + 3)} = 0$$

Solution:

The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = 0 & \rightarrow f' = 0 \\ x = 1 & \rightarrow f' = \dfrac{45}{0} = Undefined \\ x = -3 & \rightarrow f' = \dfrac{405}{0} = Undefined \end{cases}$$

Example Problem #1 – Cont'd

Find the Critical Values for the following derivative functions ($f'(x)$):

f) The derivative function ($f'(x)$):

$$f'(x) = \frac{45 \cdot x^3 \cdot (x + 4)}{(x^2 - 64)(1 + x^2)} = 0$$

Solution:

The **Critical Numbers** (CN) are:

$$CN = \begin{cases} x = 0 & \rightarrow f' = 0 \\ x = -4 & \rightarrow f' = 0 \\ x = 8 & \rightarrow f' = \dfrac{276{,}480}{0} = Undefined \\ x = -8 & \rightarrow f' = \dfrac{92{,}160}{0} = Undefined \end{cases}$$

Example Problem #2

Given the following function, find all the critical points, local mins and maximums, intervals of increase and decrease, and use them to sketch the graph of the function:

The function:

$$f(x) = \frac{x + 1}{x - 2}$$

Solution:

Step 1:

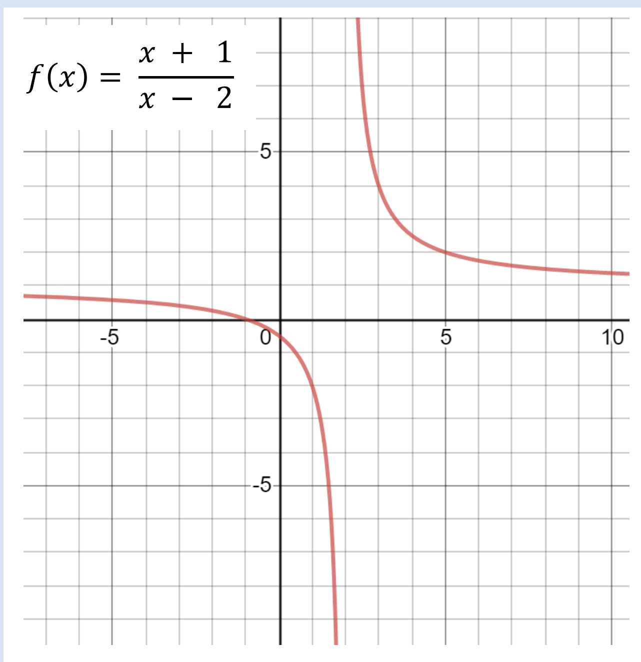
Test for **Vertical Asymptote**. The Vertical Asymptote will be one of the Critical Numbers.

$$y = f(a) = \frac{h(a)}{0} = Undefined$$

The Vertical Asymptotes (VA) are **Critical Numbers** (CN) that are undefined:

$$VA = CN = \begin{cases} x = 2 \end{cases} \rightarrow f' = \frac{3}{0} = Undefined$$

Moreover, the graph will have **vertical asymptotes** at ($x = 2$).



Example Problem #2 – Cont'd

Step 2:

Test for **Horizontal Asymptote**.

$$y = \lim_{x \to \infty} f(x) = L$$

$$y = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{x + 1}{x - 2}\right) = \lim_{x \to \infty} \left(\frac{x}{x}\right) \left[\frac{1 + \frac{1}{x}}{1 - \frac{2}{x}}\right]$$

$$y = \lim_{x \to \infty} \left[\frac{1 + \frac{1}{x}}{1 - \frac{2}{x}}\right] = \frac{1 + \frac{1}{\infty}}{1 - \frac{2}{\infty}} = \frac{1}{1} = 1$$

$$\textbf{Horizontal Asymptote} \rightarrow HA = y = 1$$

Step 3:

Find the Critical Numbers (CN), by taking the derivative of the above function, and setting equal to zero ($f'(x) = 0$).

$$f'(x) = \frac{d}{dx}\left(\frac{x + 1}{x - 2}\right)$$

You can either use the chain rule, see **left side** of equation below; or on the **right-side**, use the Quotient rule [$f'(x) = \frac{f'g - fg\prime}{g^2}$].

$$f'^{(x)} = \frac{(x - 2) \cdot \frac{d}{dx}(x + 1) - (x + 1) \cdot \frac{d}{dx}(x - 2)}{(x - 2)^2}$$

$$f'(x) = \frac{x - 2 - (x + 1)}{(x - 2)^2} = \frac{-3}{(x - 2)^2}$$

Example Problem #2 – Cont'd

Next, set the derivative equal to zero ($f'(x) = 0$):

$$f'(x) = \frac{-3}{(x - 2)^2} = 0$$

The **Critical Numbers** (CN), in this case is identical to the Vertical Asymptote, given by the following.

$$VA = CN = \{x = 2 \quad \rightarrow \quad f' = \frac{-3}{0} = Undefined$$

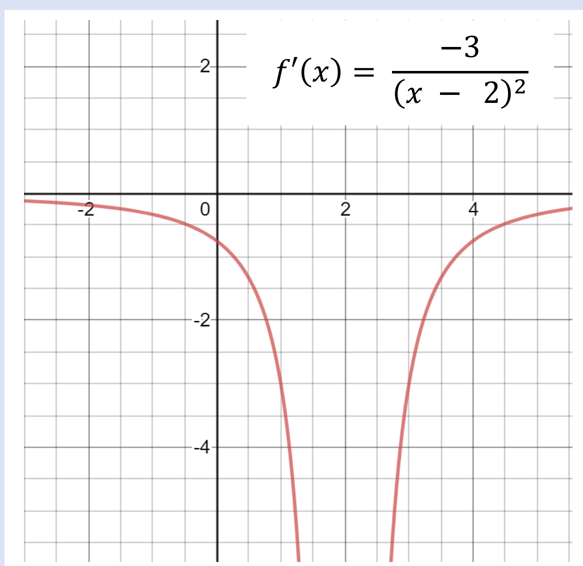
Step 4:

Find the **Critical Points** of the functions [$CP = (x, f(x))$]:

Using the Critical Numbers (CN) → ($x = 2$), substitute into the original function ($f(x)$).

$$f(x) = \frac{x + 1}{x - 2}$$

$$f(2) = \frac{-3}{0} = Undefined$$



Example Problem #2 – Cont'd

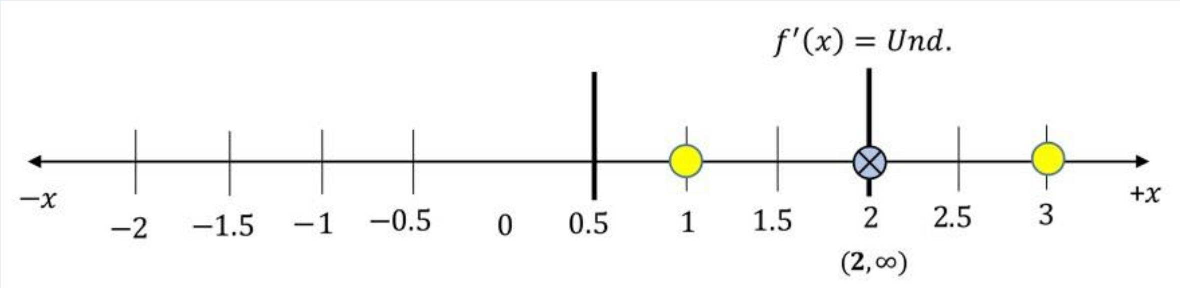
The **Critical Points** (CP), are the following.

$$CP = \{(2, f(2)) \quad = no\ point$$

Step 5:

Create a Sign Diagram or Sketch Graph, which includes Critical Numbers ($f'(x) = 0$) and artribary Test Points, along the **domain of the function**.

The Tests Points, are single arbitrary values, selected on the right-side and left-side of the Critical Numbers.



The **Test Points**, are the following, and are to be substituted into the derivative equation ($f'(x)$).

$$TP = \begin{cases} 1, f'(1) \\ 3, f'(3) \end{cases}$$

Example Problem #2 – Cont'd

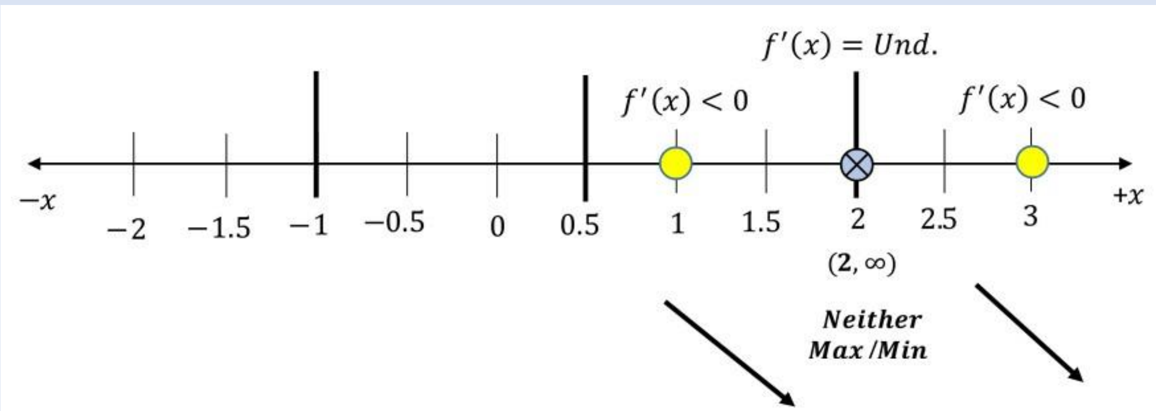
Step 7:

Create arbitrary Test Points, and substitute into the derivative of the function ($f'(x)$); where the **derivative is less than zero, the slope is decreasing** ($f'(x) < 0$), and where the **derivative is greater than zero, the slope is increasing** ($f'(x) > 0$):

$$f'(x) = \frac{-3}{(x - 2)^2}$$

$$f'(1) = \frac{-3}{(1 - 2)^2} = -\frac{3}{1} = -3 < 0 \rightarrow Decreasing$$

$$f'(3) = \frac{-3}{(3 - 2)^2} = -\frac{3}{1} = -3 < 0 \rightarrow Decreasing$$



Example Problem #2 – Cont'd

Step 8:

Find Open Intervals of Increase (OI) & Decrease (OD), and finalize results:

$$\text{Open Interval of Increase} \rightarrow OI \quad = \quad None$$

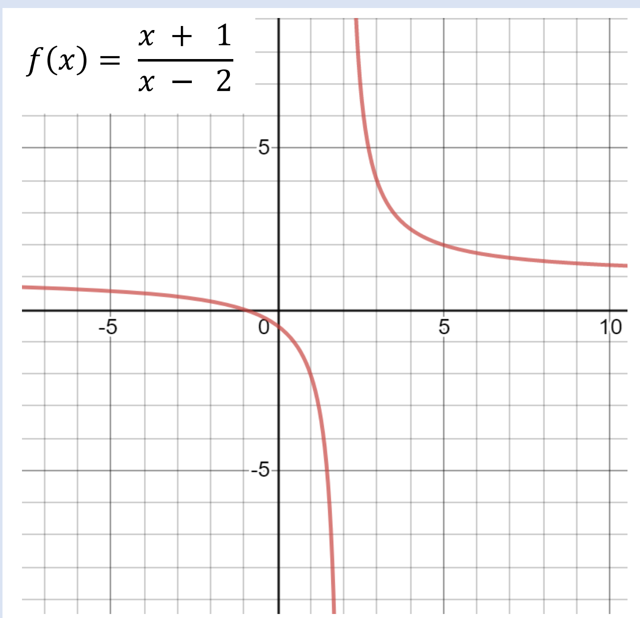
$$\text{Open Interval of Decrease} \rightarrow OD \quad = \quad (-\infty, 2)\ U\ (2, \infty)$$

The **Critical Points** (CP), are the following.

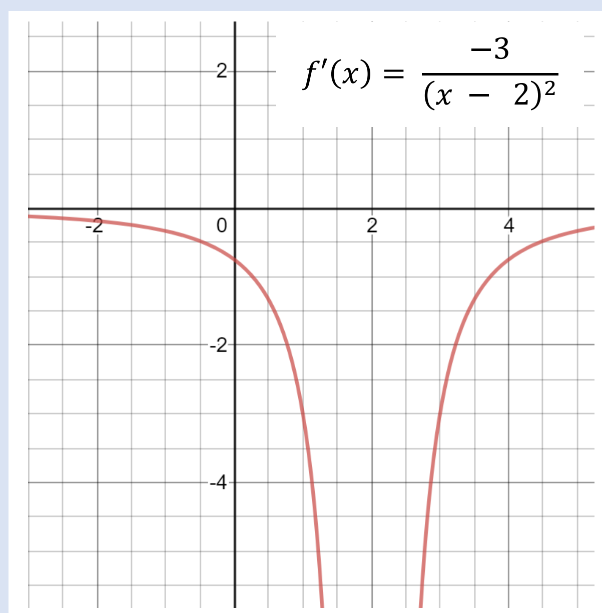
$$CP \ = \ \{(2, f(2)) \qquad = no\ point$$

$$Neither\ (Max\ nor\ Min)\ when\ \ x = 2$$

Here is the graph of the function.



Graph of the Derivative



Example Problem #3

Use the concepts from algebra to find the Vertical Asymptote and the Horizontal Asymptote information about the values of ($f(x)$) to help graph:

$$f(x) = \frac{x + 2}{2x^2 + 7x - 4}$$

Solution:

First, let's factor the denominator of this function:

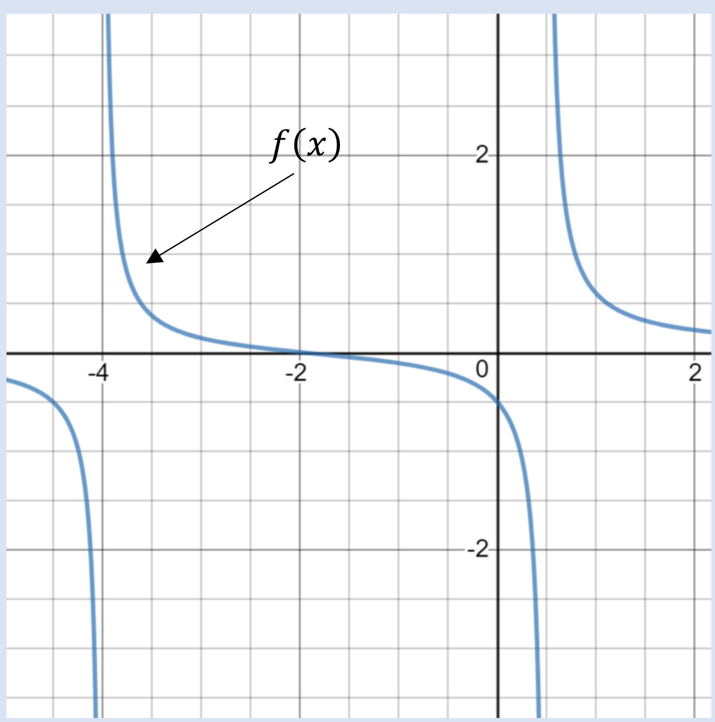
$$f(x) = \frac{x + 2}{(2x - 1)(x + 4)}$$

Since the denominator of a fraction cannot equal to 0, the x-values cannot equal to (1/2) or (-4), hence the domain of this function is D = $\mathbb{R} \setminus \{-4, \frac{1}{2}\}$.

Moreover, the graph will have **vertical asymptotes** at ($x = -4$) and ($x = \frac{1}{2}$).

Since the degree of the numerator (<) the degree of the denominator,

the line ($y = 0$) is a horizontal asymptote.



Example Problem #3 – Cont'd

Test for **Horizontal Asymptote**

$$y = f(a) = \frac{h(a)}{0} = Undefined$$

$$f(x) = \frac{x + 2}{2x^2 + 7x - 4} = \frac{x + 2}{(2x - 1)(x + 4)}$$

Next, substitute critical values ($x = -4$) and ($x = \frac{1}{2}$), into the above function.

$$f\left(\frac{1}{2}\right) = \frac{\frac{1}{2} + 2}{\left(2\left(\frac{1}{2}\right) - 1\right)\left(\frac{1}{2} + 4\right)} = \frac{\frac{5}{2}}{(0)\left(\frac{9}{2}\right)} = \frac{2.5}{0} = undefined$$

$$f(-4) = \frac{-4 + 2}{(2(-4) - 1)(-4 + 4)} = \frac{-2}{(-9)(0)} = \frac{-2}{0} = undefined$$

$$\boldsymbol{Vertical\ Asymptote} \rightarrow VA = \begin{cases} x = \frac{1}{2} \\ x = -4 \end{cases}$$

It is enough to use the degree comparison to determine the Horizontal Asymptotes, but here is also for you the calculus way of determining the HA.

Test for **Horizontal Asymptote**

$$y = \lim_{x \to \infty} f(x) = L$$

$$y = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x + 2}{2x^2 + 7x - 4} = \lim_{x \to \infty} \left(\frac{x}{x^2}\right)\left[\frac{1 + \frac{2}{x}}{2 + \frac{7x}{x^2} - \frac{4}{x^2}}\right]$$

$$y = \lim_{x \to \infty} \left(\frac{1}{x}\right)\left[\frac{1 + \frac{2}{x}}{2 + \frac{7}{x} - \frac{4}{x^2}}\right] = \lim_{x \to \infty} \left(\frac{1}{\infty}\right)\left[\frac{1 + \frac{2}{\infty}}{2 + \frac{7}{\infty} - \frac{4}{(\infty)^2}}\right] = \frac{1}{\infty} = 0$$

$$\boldsymbol{Horizontal\ Asymptote} \rightarrow HA = y = 0$$

Example Problem #3 – Cont'd

Let's find the derivative.

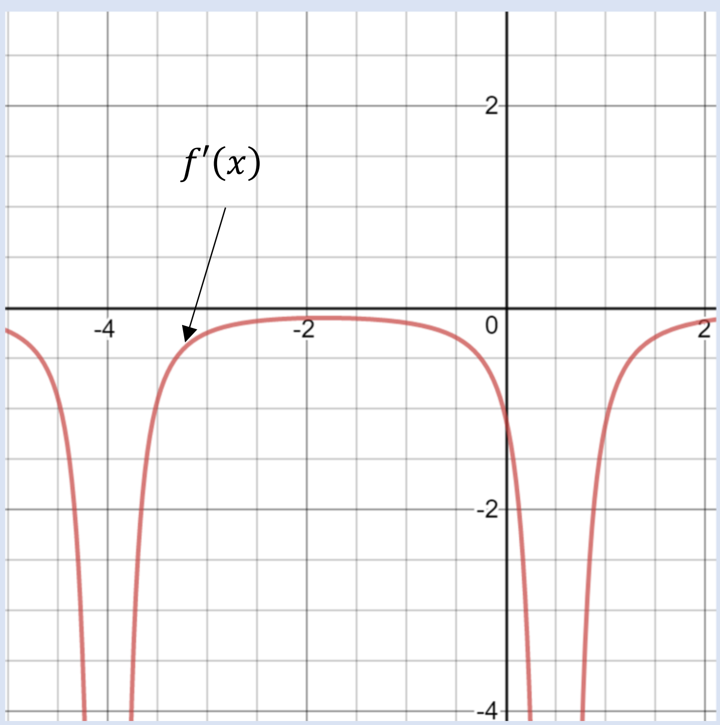
$$f'(x) = \frac{d}{dx}\left[\frac{x + 2}{2x^2 + 7x - 4}\right] = \frac{1(2x^2 + 7x - 4) - (4x+7)(x+2)}{(2x^2 + 7x - 4)^2}$$

$$f'(x) = \frac{-2x^2 - 8x - 18}{(2x^2 + 7x - 4)^2} = \frac{-2x^2 - 8x - 18}{\left((2x-1)(x+4)\right)^2}$$

To find the critical numbers we need to find where ($f'(x)$) is equal to 0 or is undefined.

$f'(x) = 0$ or where $f'(x) = Undefined.$

Thus, in the above derivative function ($f'(x)$), there are two types of critical values.



Example Problem #3 – Cont'd

The first, critical value occurs when the derivative ($f'(x) = 0$), that is when the numerator of the fraction equals 0.

$$(-2x^2 - 8x - 18) = -2(x^2 + 4x + 9) = 0$$

$$x = \frac{-4 \pm \sqrt{16 - 4(9)}}{2} = \frac{-4 \pm \sqrt{-20}}{2} = -2 \pm \sqrt{-5}$$

Solving the quadratic factor by the quadratic formula, we quickly see that we get a negative quantity under the radical, hence we get no real solutions.

$$f'(x) = 0 = No\ Real\ Solutions$$

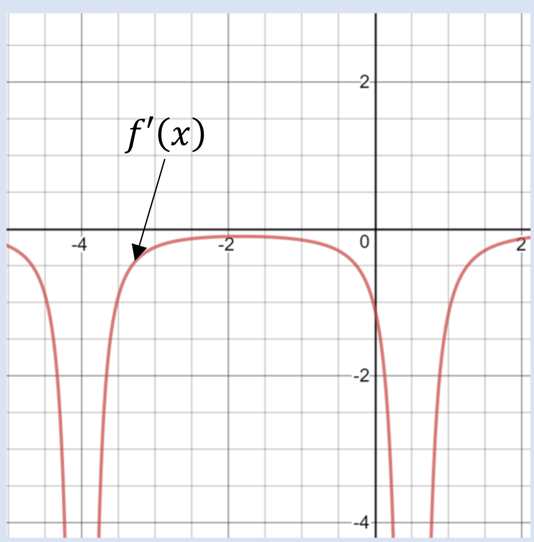
Moreover, $-2x^2 - 8x - 18$ is always a negative quantity as it is a downward looking parabola with no x-axis intercepts, as we just found out.

The second, critical value occurs when the derivative ($f'(x) = Undefined$), that is when the denominator of the fraction equals ($\frac{1}{0} = Undefined$)

The denominator of $f'(x)$, is always a positive quantity, since it is being squared.

$$\frac{1}{(2x^2 + 7x - 4)^2} = \frac{1}{\left((2x-1)(x+4)\right)^2} = \frac{1}{0} = Undefined$$

Hence ($f'(x) < 0$) on the whole domain, so ($f(x)$) is a decreasing of the whole domain. See graph and table below.



Example Problem #3 – Cont'd

Next, choose Test Points around the Critical Numbers (CN).

$$CN = \begin{cases} x = -4 \\ x = 1/2 \end{cases}$$

x	$f'(x) = \dfrac{-2x^2 - 8x - 18}{\left((2x-1)(x+4)\right)^2}$	$f'(x) \rightarrow$ Increase/Decrease
-5	$f'(-5) = \dfrac{-2(-5)^2 - 8(-5) - 18}{\left((2(-5)-1)((-5)+4)\right)^2} = -0.2314$	$f'(-5) < 0 \rightarrow Decreasing$
-4	$f'(-4) = Undefined$	$f'(-4) = Undefined \rightarrow Neither\ Local\ Max/Min$
0	$f'(0) = \dfrac{-2(0)^2 - 8(0) - 18}{\left((2(0)-1)((0)+4)\right)^2} = -1.125$	$f'(0) < 0 \rightarrow Decreasing$
1/2	$f'(1/2) = Undefined$	$f'(1/2) = Undefined \rightarrow Neither\ Local\ Max/Min$
1	$f'(1) = \dfrac{-2(1)^2 - 8(1) - 18}{\left((2(1)-1)((1)+4)\right)^2} = -1.12$	$f'(1) < 0 \rightarrow Decreasing$

Hence ($f'(x) < 0$) on the whole domain, so ($f(x)$) is a decreasing of the whole domain.

Example Problem #3 – Cont'd

Putting all the information together, we have

$$\textbf{Vertical Asymptote} \rightarrow VA = \begin{cases} x = \dfrac{1}{2} \\ x = -4 \end{cases}$$

$$\textbf{Horizontal Asymptote} \rightarrow HA = y = 0$$

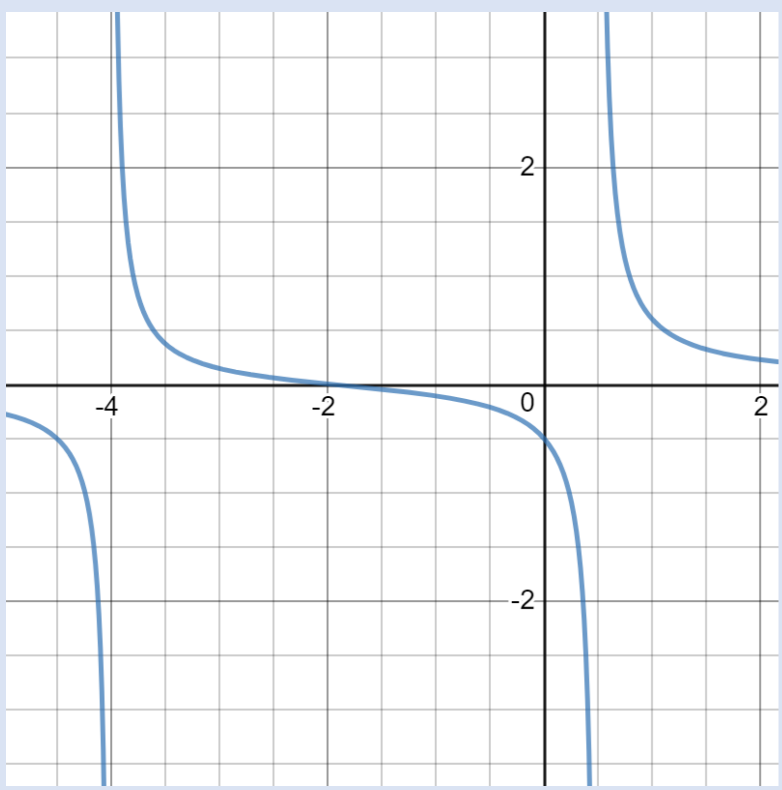
$f(x)$ is decreasing on the whole domain.

The intercepts of $f(x)$ are:

y - intercept: $If \quad x = 0,\ y = -\frac{1}{2},$ so $(0, -\frac{1}{2})$

x - intercept: $If \quad y = 0,\ x + 2 = 0\,,\ x = -2,$ so $(-2, 0)$

Here is the graph of the function.



Example Problem #4

Given the following function, find all the critical points, local mins and maximums, intervals of increase and decrease, and use them to sketch the graph of the function:

The function:

$$f(x) = \frac{14}{x^2 - x - 2}$$

Solution:

Step 1:

First, let's factor the denominator of this function ($f(x)$):

$$f(x) = \frac{14}{(x + 1)(x - 2)}$$

Step 2:

Test for **Vertical Asymptote**. The Vertical Asymptote will be one of the Critical Numbers.

$$y = f(a) = \frac{h(a)}{0} = \infty = Undefined$$

The Vertical Asymptotes (VA) are **Critical Numbers** (CN) that are undefined:

$$VA = CN = \begin{cases} x = -1 \rightarrow f' = \frac{14}{0} = Undefined \\ x = 2 \rightarrow f' = \frac{14}{0} = Undefined \end{cases}$$

Moreover, the graph will have **vertical asymptotes** at ($x = -1$) and ($x = 2$).

Example Problem #4 – Cont'd

Step 3:

Test for **Horizontal Asymptote**.

$$y = \lim_{x \to \infty} f(x) = L$$

$$y = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{14}{x^2 - x - 2} = \lim_{x \to \infty} \left(\frac{1}{x^2}\right)\left[\frac{14}{1 - \frac{x}{x^2} - \frac{2}{x^2}}\right]$$

$$y = \lim_{x \to \infty} \left(\frac{1}{x^2}\right)\left[\frac{14}{1 - \frac{1}{x} - \frac{2}{x^2}}\right] = \lim_{x \to \infty} \left(\frac{1}{\infty}\right)\left[\frac{14}{1 - \frac{1}{\infty} - \frac{2}{(\infty)^2}}\right] = \frac{1}{\infty} = 0$$

$$\textbf{\textit{Horizontal Asymptote}} \rightarrow HA = y = 0$$

Step 4:

Find the Critical Numbers (CN), by taking the derivative of the above function, and setting equal to zero ($f'(x) = 0$).

$$f'(x) = \frac{d}{dx}\left[\frac{14}{x^2 - x - 2}\right]$$

You can either use the chain rule, see **left side** of equation below; or on the **right-side**, use the Quotient rule [$f'(x) = \frac{f'g - fg'}{g^2}$].

$$f'(x) = \frac{d}{dx}[14 \cdot (x^2 - x - 2)^{-1}] = \frac{(x^2 - x - 2) \cdot \frac{d}{dx}(14) - 14 \cdot \frac{d}{dx}(x^2 - x - 2)}{(x^2 - x - 2)^2}$$

$$f'(x) = \frac{-14 \cdot (2x - 1)}{(x^2 - x - 2)^2} = \frac{-14 \cdot (2x - 1)}{\big((x + 1)(x - 2)\big)^2}$$

Example Problem #4 – Cont'd

Next, set the derivative equal to zero ($f'(x) = 0$):

$$f'(x) = \frac{-14 \cdot (2x - 1)}{\big((x + 1)(x - 2)\big)^2} = 0$$

The **Critical Numbers** (CN), are the following.

$$CN = \begin{cases} x = \frac{1}{2} & \rightarrow f' = 0 \\ x = -1 & \rightarrow f' = \frac{42}{0} = Undefined \\ x = 2 & \rightarrow f' = \frac{-42}{0} = Undefined \end{cases}$$

Step 5:

Find the **Critical Points** of the functions [$CP = \big(x, f(x)\big)$]:

Using the Critical Numbers (CN) → ($x = \frac{1}{2}$), ($x = -1$), and ($x = 2$), substitute into the original function ($f(x)$).

$$f(x) = \frac{14}{(x + 1)(x - 2)}$$

$$f\left(\frac{1}{2}\right) = \frac{14}{\left(\frac{1}{2} + 1\right)\left(\frac{1}{2} - 2\right)} = -\frac{56}{9} = -6.22$$

$$f(-1) = \frac{42}{0} = Undefined$$

$$f(2) = \frac{-42}{0} = Undefined$$

Example Problem #4 – Cont'd

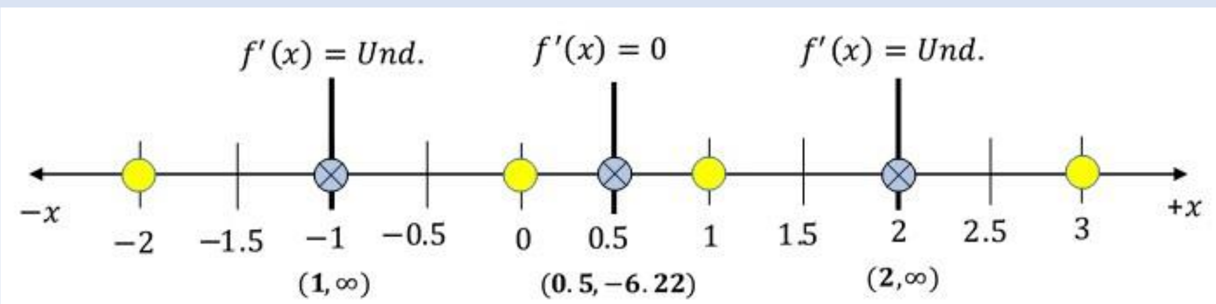
The **Critical Points** (CP), are the following.

$$CP = \begin{cases} \left(\frac{1}{2}, f(\frac{1}{2})\right) & = \left(\frac{1}{2}, -6.22\right) \\ (-1, f(-1)) & = \ no\ point \\ (2, f(2)) & = \ no\ point \end{cases}$$

Step 6:

Create a Sign Diagram or Sketch Graph, which includes Critical Numbers ($f'(x) = 0$) and artribary Test Points, along the **domain of the function**.

The Tests Points, are single arbitrary values, selected on the right-side and left-side of the Critical Numbers.



The **Test Points**, are the following, and are to be substituted into the derivative equation ($f'(x)$).

$$TP = \begin{cases} -2, f'(-2) \\ 0, f'(0) \\ 1, f'(1) \\ 3, f'(3) \end{cases}$$

Example Problem #4 – Cont'd

Step 7:

Create arbitrary Test Points, and substitute into the derivative of the function ($f'(x)$); where the **derivative is less than zero, the slope is decreasing** ($f'(x) < 0$), and where the **derivative is greater than zero, the slope is increasing** ($f'(x) > 0$):

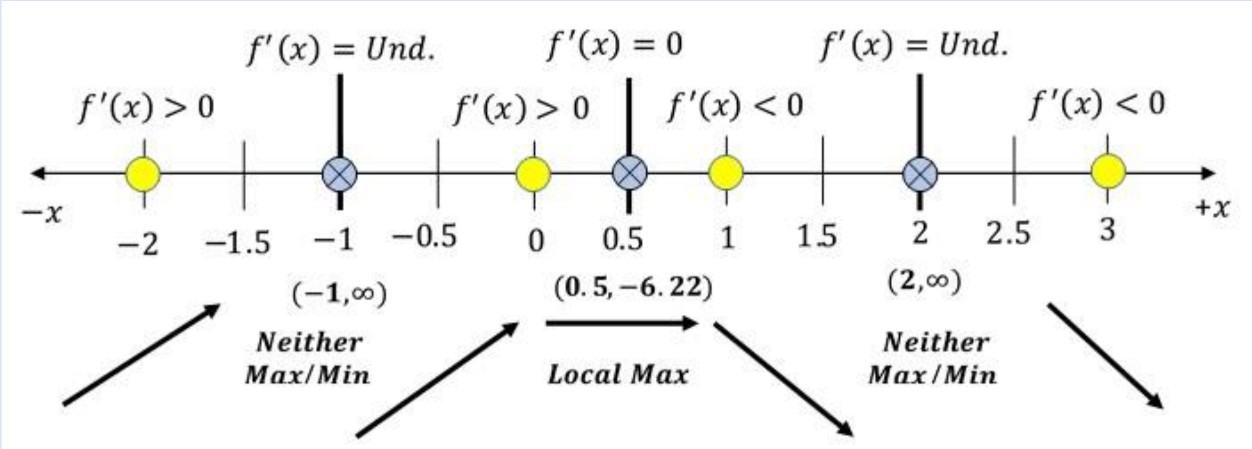
$$f'(x) = \frac{-14 \cdot (2x - 1)}{\left((x + 1)(x - 2)\right)^2}$$

$$f'(-2) = \frac{-14 \cdot (2(-2) - 1)}{\left((-2 + 1)(-2 - 2)\right)^2} = \frac{35}{8} = 4.375 > 0 \rightarrow Increasing$$

$$f'(0) = \frac{-14 \cdot (2(0) - 1)}{\left((0 + 1)(0 - 2)\right)^2} = \frac{7}{2} = 3.5 > 0 \rightarrow Increasing$$

$$f'(1) = \frac{-14 \cdot (2(1) - 1)}{\left((1 + 1)(1 - 2)\right)^2} = -\frac{7}{2} = -3.5 < 0 \rightarrow Decreasing$$

$$f'(3) = \frac{-14 \cdot (2(3) - 1)}{\left((3 + 1)(3 - 2)\right)^2} = -\frac{35}{8} = -4.375 < 0 \rightarrow Decreasing$$



Example Problem #4 – Cont'd

Step 8:

Find Open Intervals of Increase (OI) & Decrease (OD), and finalize results:

$$\text{Open Interval of Increase} \rightarrow OI \quad = \quad (-\infty, -1)\ U\ (1, 0.5)$$

$$\text{Open Interval of Decrease} \rightarrow OD \quad = \quad (0.5, 2)\ U\ (2, \infty)$$

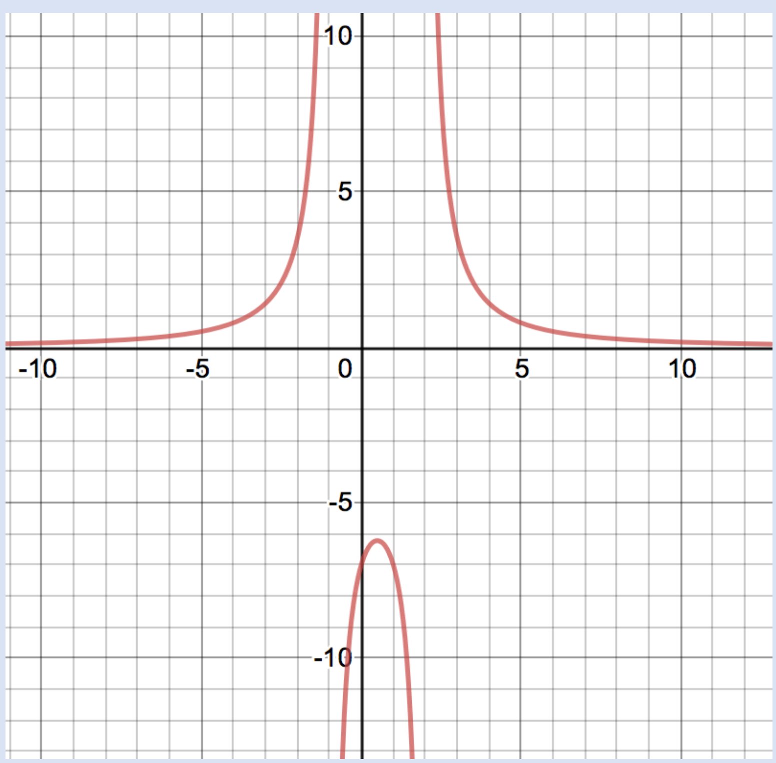
The **Critical Points** (CP), are the following.

$$CP \;=\; \begin{cases} (0.5, f(0.5)) & = \ (0.5, -6.22) \\ (-1, f(-1)) & = no\ point \\ (2, f(2)) & = \ no\ point \end{cases}$$

$$Local\ Maximum \quad = \quad (0.5, -6.22)$$

$$Neither\ (Max\ nor\ Min)\ when\ x = \ -1\ or\ x = 2$$

Here is the graph of the function.



2.2 - EXERCISES

Find the critical numbers and then the critical points of the functions.			
1.	$y = \frac{x}{x-3}$	2.	$f(x) = \frac{3x^2}{x^2-4}$
3.	$f(x) = \frac{x-1}{x+3}$	4.	$f(x) = \frac{5}{x-2}$

Sketch the graph of the following functions, by finding the local mins and maximums and the intervals of increase and decrease and making a sign diagram for the first derivative.			
5.	$f(x) = \frac{5}{x-2}$	6.	$f(x) = \frac{3x^2}{x^2-4}$
7.	$f(x) = \frac{x-4}{x^2-2x-15}$	8.	$f(x) = \frac{x-1}{x+3}$
9.	$f(x) = \frac{x}{x^2-9}$	10.	$f(x) = \frac{3x}{x^2+1}$

11.	Suspose that the cost of producing a certain product is \$10/unit and that the fixed costs are \$200. The cost function would be $$C(x) = 10x + 200.$$ a) Find the average cost function $AC(x)$ and graph it for $x > 0$. b) What happens to this graph as x gets larger and larger, i.e. what value does the average cost get close to as x approaches infinity?

Solutions:

1. Critical numbers: x = 3, Critical points none

2. Critical numbers: x = 0, x = 2, x = -2, Critical points (0, 0)

3. Critical numbers: x = -3, Critical points none

4. Critical numbers: x = 2, Critical points none

5. No local min or local max, decrease on (-∞, ∞)

6. Local max (0, 0), increase $(-\infty, -2) \cup (-2, 0)$,
   decrease on (0, 2) ∪ (2, ∞)

7. No local min or local max, decrease on (-$\infty, -3$) U $(-3, 5)$ U $(5, \infty)$

8. No local min or local max, increase on (-$\infty, -3), (-3, \infty)$

9. No local min or local max, decrease on (-$\infty, -3$) U $(-3, 3)$ U $(3, \infty)$

10. Local min (-1, -1.5), Local max (1, 1.5), increase on (-1, 1),
    decrease (-$\infty, -1$), (1, ∞)

11. $AC(x) = 10 + 200/x$. As x gets larger and larger the average cost per item approaches $10/item, which is the same as the cost per item.

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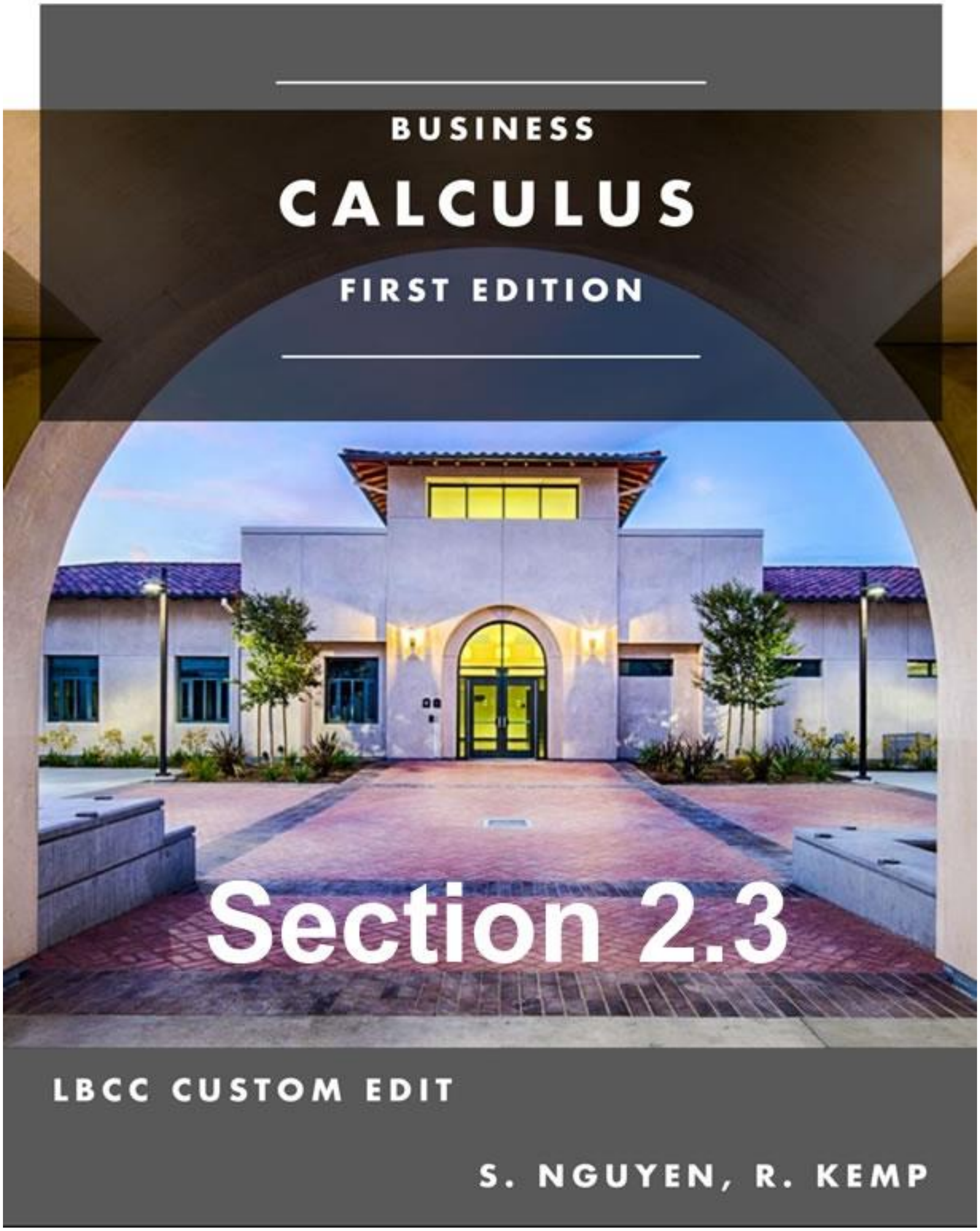
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Section 2.3
LBCC CUSTOM EDIT
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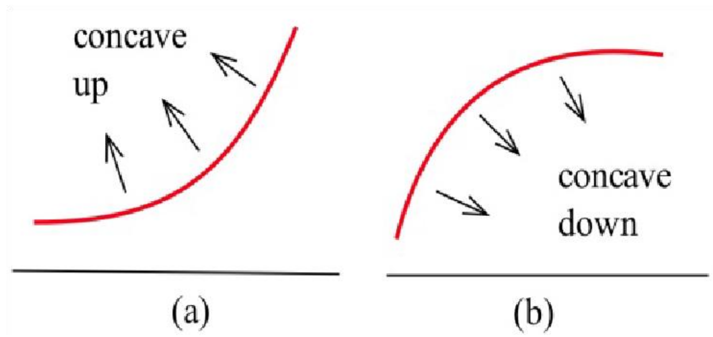
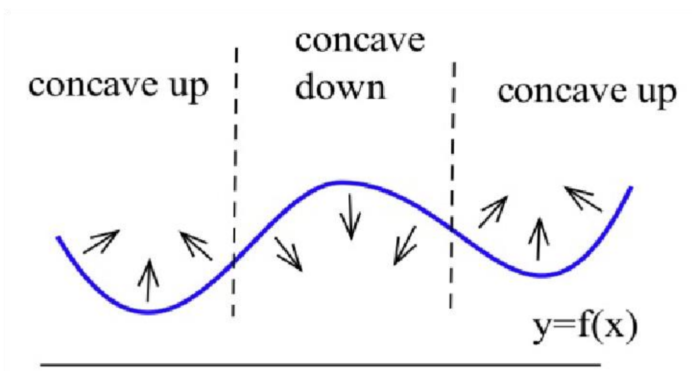
2.3 - THE SECOND DERIVATIVE. INFLECTION POINTS. CONCAVITY

Graphically, a function is **concave up** if its graph is curved with the opening upward (a in the figure).

Similarly, a function is **concave down** if its graph opens downward (b in the figure).



This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.

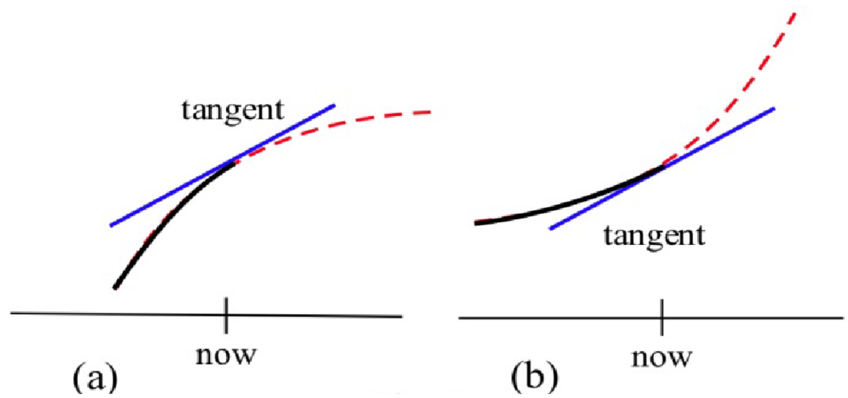


For example, **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed.

In the figure below, (f) is the number of people who have the disease at time x, and two different situations are shown.

In both (a) and (b), the number of people with the disease, ($f(now)$), and the rate at which new people are getting sick, ($f'(now)$), are the same.

The difference in the two situations is the concavity of (f), and that difference in concavity might have a big effect on your decision.



In (a), (f) is concave down at "now", the slopes are decreasing, and it looks as if it's tailing off. We can say "(f') is increasing at a decreasing rate."

It appears that the current methods are starting to bring the epidemic under control.

In (b), (f) is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the *derivative* is increasing or decreasing.

The derivative of a function f is a function that gives information about the slope off. **The derivative tells us if the original function is increasing or decreasing.**

Because (f') is a function, we can take its derivative. This second derivative (f'') also gives us information about our original function (f).

The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down.**

Second Derivative

The function (f)

$$y = f(x)$$

The **second derivative of** (f) is the derivative of ($y' = f'(x)$):

$$y'' = f''(x) = \frac{d}{dx}f'(x) = \frac{d^2}{dx^2}f(x)$$

$$y'' = \frac{d}{dx}y' = \frac{d^2}{dx^2}y$$

Using prime notation, this is *($f''(x)$)* or *(y'')*.

If ($f''(x) > 0$) **is positive** on an interval, the graph of ($y = f(x)$) is **concave up** on that interval**.** We can say that *f* is increasing (or decreasing) **at an increasing rate**.

If ($f''(x) < 0$) **is negative** on an interval, the graph of ($y = f(x)$) is **concave down** on that interval**.** We can say that *f* is increasing (or decreasing) **at a decreasing rate.**

- A curve that curls upward is said to be concave up, and a curve that curls downward is said to be concave down.
- A point where the concavity *changes* (from up to down or down to up) is called an inflection point.

Inflection Points

Definition:

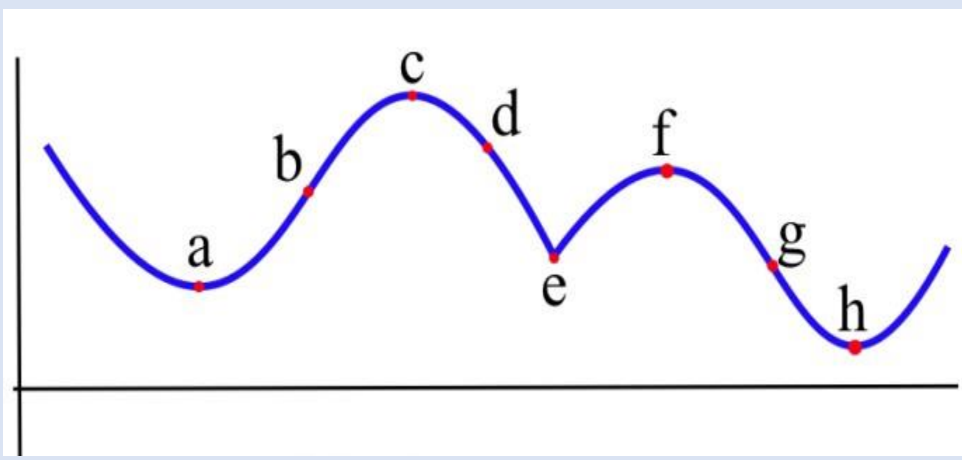
An **Inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

Working Definition:

An **inflection point** is a point on the graph where the second derivative changes sign.

Example Problem #1

Which of the labeled points in the graph below are inflection points?



Solution:

The concavity changes at points (b) and (g), therefore there are inflection points.

At points (a) and (h), the graph is concave up on both sides and at points (c), and (f) the graph is concave down on both sides, so the concavity does not change at those points. Therefore these are not inflections points.

At the point (e) there is no change in the concavity, therefore this is not an inflection point.

The Inflection points are (b) and (g); at these points the concavity changes.

Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

For the second derivative to change signs, it must either be zero or be undefined.

So, to find the inflection points of a function we only need to check the points where ($f''(x)$) is equal to 0 or is undefined.

Definition

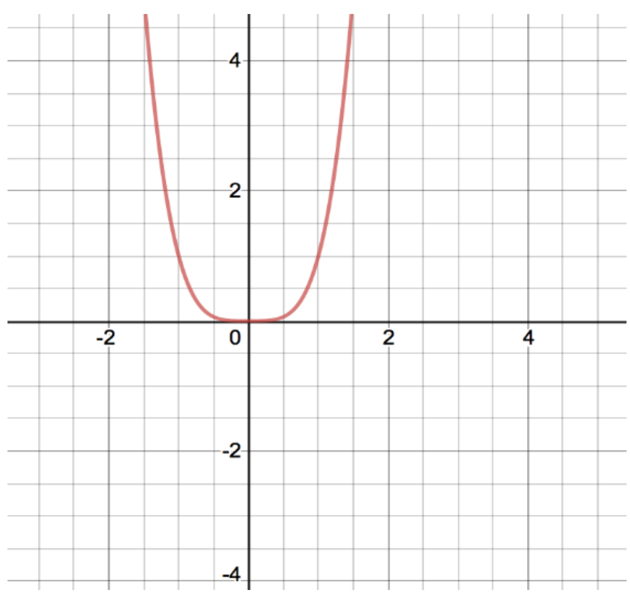
An **Inflection point** for a function (f) can be found where either

$$f''(a) = 0 \quad \text{or} \quad f''(a) = Undefined$$

For a point $(a, f(a))$ to be an inflection point, f" must change signs left and right of a.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of (f'') changes sign.

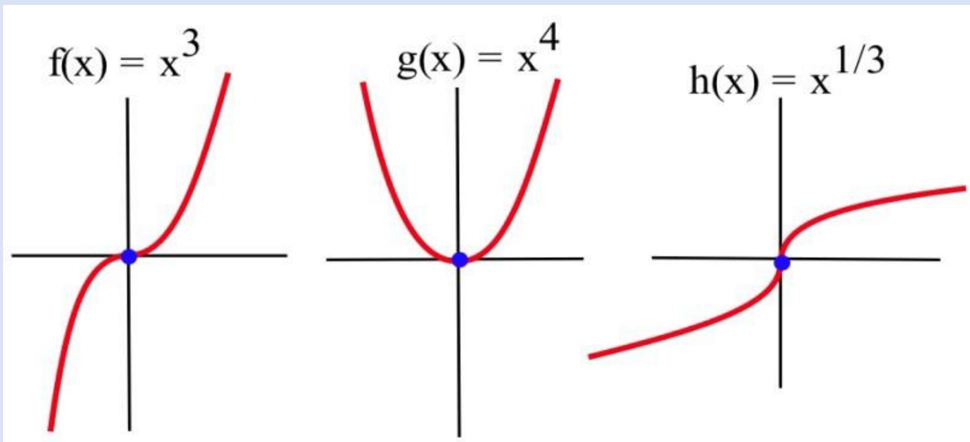
Here is the graph of the $f(x) = x^4$ function. The second derivative is equal to 0 when x = 0, $f"(0) = 0$, yet the function is not changing convcavity at x = 0, as you can see.



Example Problem #2

Here are the graphs of $f(x) = x^3$, $g(x) = x^4$, and $h(x) = x^{\frac{1}{3}}$.

For which of these functions is (0, 0) an inflection point?



Solution:

Graphically, it is clear that only (f) and (h) change concavity at (0, 0).

So (0, 0) is an inflection point for functions (f) and (h), but not (g).

The function (g) does not change concavity.

Let's look what happens to (f) algebraically.

$$f'(x) = \frac{d}{dx}f(x) = 3x^2 \quad \text{and} \quad f''(x) = \frac{d}{dx}f'(x) = 6x$$

The only points at which ($f''(x) = 0$), is at ($x = 0$).

If ($x < 0$), ($f''(x) < 0$) so, (f) is **concave down**;

If ($x > 0$), ($f''(x) > 0$) so, (f) is **concave up**.

Function (f) changes concavity at ($x = 0$), so (0, 0) is an inflection point.

Example Problem #2 – Cont'd

Let's look what happens to (g) algebraically.

$$g'(x) = \frac{d}{dx}g(x) = 4x^3 \quad \text{and} \quad g''(x) = \frac{d}{dx}g'(x) = 12x^2$$

The only points at which $(g''(x) = 0)$, is at $(x = 0)$.

If $(x < 0)$, $(g''(x) > 0)$ so, (g) is **concave up**;

If $(x > 0)$, $(g''(x) > 0)$ so, (g) is **concave up**.

Function (g) does not changes concavity at $(x = 0)$, so (0, 0) is not an inflection point.

Let's look what happens to (h) algebraically.

$$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) \quad \text{and} \quad h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$$

$$h'(x) = \frac{1}{3}x^{\left(\frac{1}{3}-1\right)} \quad \text{and} \quad h''(x) = \frac{1}{3}\left(-\frac{2}{3}x^{\left(-\frac{2}{3}-1\right)}\right)$$

$$h'(x) = \frac{1}{3}x^{-\frac{2}{3}} \quad \text{and} \quad h''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$

The point at which $(h''(x) = Undefined)$, is at $(x = 0)$.

If $(x < 0)$, $(h''(x) > 0)$ so, (h) is **concave up**;

If $(x > 0)$, $(h''(x) < 0)$ so, (h) is **concave down**.

Function (h) changes concavity at $(x = 0)$, so (0, 0) is not an inflection point.

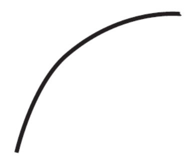
Sketching without an Equation

Of course, graphing calculators and computers are great at graphing functions. Calculus provides a way to illuminate what may be hidden or out of view when we graph using technology.

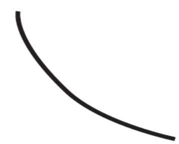
More importantly, calculus gives us a way to look at the derivatives of functions for which there is no equation given. Below we sketch the curve of function (f), by estimating the first derivatives as increasing or decreasing “slopes” (f') on the curves, and the second derivatives as the “curl” or “Concavity” (f'') of the curve.

3. ***Increasing*** and **concave *up*** ($f' > 0, f'' > 0$)

4. ***Increasing*** and **concave *down*** ($f' > 0, f'' < 0$)



5. ***Decreasing*** and **concave *up*** ($f' < 0, f'' > 0$)



6. ***Decreasing*** and **concave down** ($f' < 0, f'' < 0$)

Summary of Derivative Information about the Graph

$f(x)$	**Increasing & Concave Up**	**Increasing & Concave Down**	**Decreasing & Concave Up**	**Decreasing & Concave down**
$f'(x)$	**+**	**+**	**-**	**-**
$f''(x)$	**+**	**-**	**+**	**-**

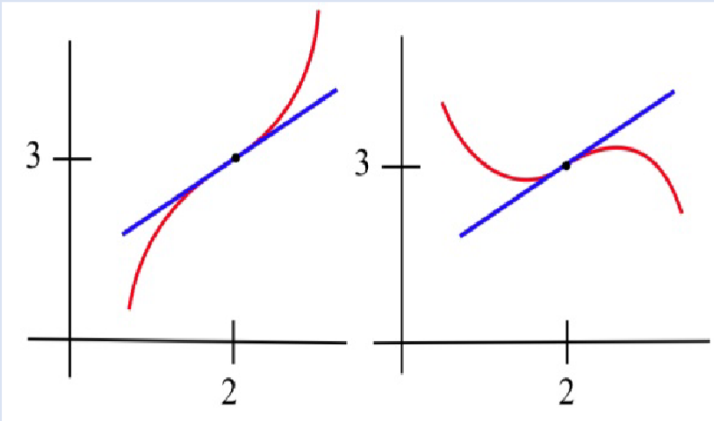
When ($f'(x) = 0$), the graph of ($f(x)$) may have a local max or min.

When ($f''(x) = 0$), the graph of ($f(x)$) may have an inflection point.

Example Problem #3

Sketch the graph of a function such that ($f(2) = 3$), ($f'(2) = 1$) and has an inflection point at ($IP = (2, 3)$)

Solution:

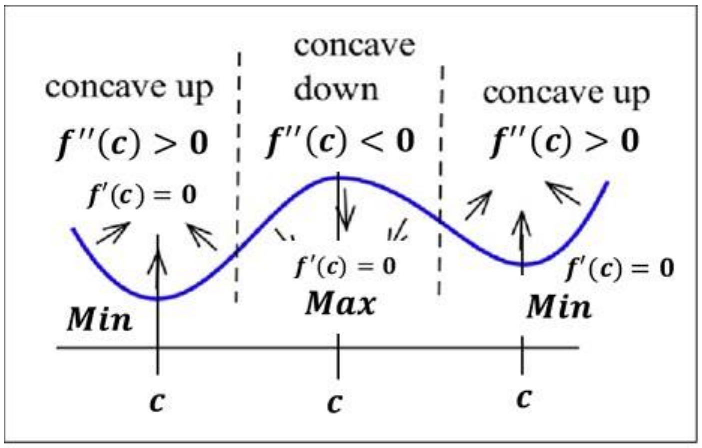


(f'') and Extreme Values of (f)

Second Derivative Test for Extreme (Max. & Min.) Points

The concavity of a function can also help us determine whether a critical point is a maximum, minimum, or neither, extreme value.

For example, if a point is at the bottom of a **concave up** function, then the point is a **local or relarive minimum value**. If a point is at the top of a **concave down** function, then the point is a **local or relative maximum value**.



The Second Derivative Test for Extremes:

Find all critical points of (f).

For those critical points where $(f'(c) = 0)$, find $(f''(c))$.

(a) If $(f''(c) < 0)$ then (f) is **concave down** and has a **local maximum** at $(x = c)$.

(b) If $(f''(c) > 0)$ then (f) is **concave up** and has a **local minimum** at $(x = c)$.

(c) If $(f''(c) = 0)$ then the test is "inconclusive", (f) may have a **local maximum**, **local minimum**, or **neither** at $(x = c)$.

The cartoon faces can help you remember the **Second Derivative Test**.

Example Problem #4

The function:

$$f(x) = 2x^3 - 15x^2 + 24x - 7$$

has Critical Numbers $(CN) \rightarrow (x = 1)$ and $(x = 4)$.

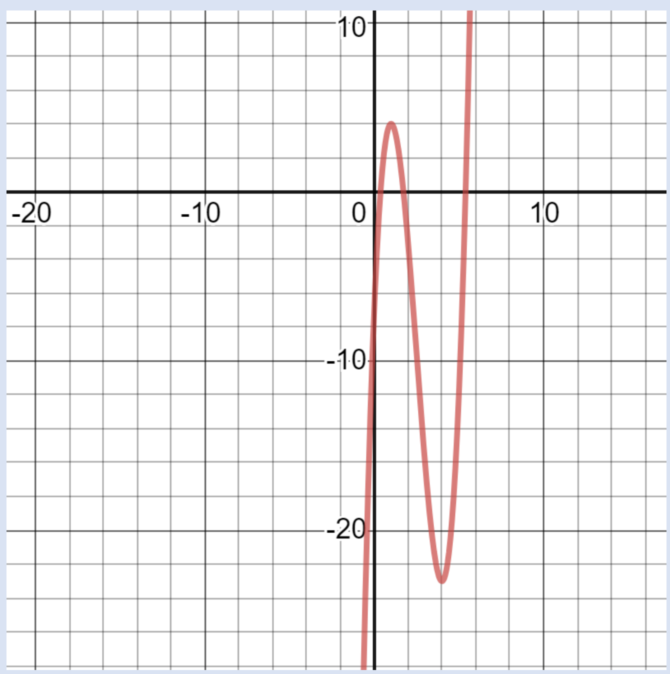
Use the second derivative test for extremes to determine if the point $(1, f(1))$ and $((4, f(4))$ are local minimums or maximums.

Solution:

We start by evaluating $(f(x))$ at $(x = 1)$ and $(x = 4)$.

$$y = f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 7 = 4$$

$$y = f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 7 = -23$$



The Critical Points are the following:

$$CP = \begin{cases} (1, f(1)) = (1, 4) \\ (4, f(4)) = (4, -23) \end{cases}$$

Example Problem #4 – Cont'd

We need to find the second derivative. We start by finding the first derivative.

$$f'(x) = \frac{d}{dx}(2x^3 - 15x^2 + 24x - 7)$$

$$f'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$$

Next we find the second derivative.

$$f''(x) = \frac{d}{dx}(6x^2 - 30x + 24)$$

$$f''(x) = 12x - 30 = 6(2x-5)$$

Now we just need to evaluate ($f''(x)$) at ($x = 1$) and ($x = 4$).

$$f''(1) = 12x - 30 = 12(1) - 30 = -18 < 0$$

So, there is a **local maximum** at: $x = 1$, $(1, 4)$

$$f''(4) = 12x - 30 = 12(4) - 30 = 18 > 0$$

So, there is a **local minimum** at: $x = 4$, $(4, -23)$

Many students like the Second Derivative Test. The Second Derivative Test is often easier to use than the First Derivative Test.

You only have to find the sign of one number for each critical number rather than two.

And if your function is a polynomial, its second derivative will probably be a simpler function than the derivative.

However, if you needed a product rule, quotient rule, or chain rule to find the first derivative, finding the second derivative can be a lot of work.

Also, even if the second derivative is easy, the Second Derivative Test doesn't always give an answer.

The First Derivative Test will always give you an answer.

Use whichever test you want to. But remember – you have to do some test to be sure that your critical point actually is a local max or min.

Second Derivative Information

Until now, we've only used first derivative information, but we could also use information from the second derivative to provide more information about the shape of the function.

Example Problem #5

Graph the function $f(x) = \sqrt[3]{x^2} = x^{2/3}$

Solution:

$f'(x) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$; This is undefined at x = 0

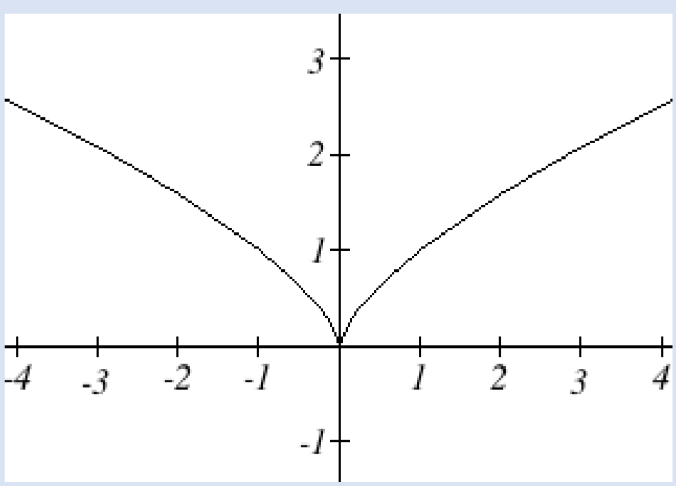
$f''(x) = \frac{d}{dx}\left(\frac{2}{3}x^{-1/3}\right) = -\frac{2}{9}x^{-4/3} = -\frac{2}{9\sqrt[3]{x^4}}$; This is also undefined at (x = 0)

These equations create two intervals we need to look at: (x < 0), and (x > 0)

On the interval (x < 0), we could test a value, say ($x = -1$) and find out that $f'(-1) = -\frac{2}{3} < 0$ and $f''(-1) = -\frac{2}{9} < 0$, so the function ($f(x)$) is decreasing, the slopes are negative, and is concave down on this interval.

On the interval (x > 0), we could test a value, say ($x = 1$) and find out that $f'(1) = \frac{2}{3} > 0$ and $f''(1) = -\frac{2}{9} < 0$, so the function ($f(x)$) is increasing, the slopes are positive, and is concave down on this interval.

We can also calculate one point on the graph of ($f(x)$), say ($f(0) = 0$), which will give us one point we can then plot.



Example Problem #6

Given the following function, find all the critical points, inflection points, local mins and maximums, intervals of increase and decrease, concavity intervals and use them to sketch the graph of the function:

The function:

$$f(x) = x^3 \cdot \left(\frac{3}{2}x - 4\right)$$

Solution:

Step 1:

Find the Critical Numbers (CN), by taking the derivative of the above function, and setting equal to zero ($f'(x) = 0$).

$$f'(x) = \frac{d}{dx}\left[x^3 \cdot \left(\frac{3}{2}x - 4\right)\right]$$

You can either use algebra to simplify, see **left side** of equation below; or on the **right-side**, use the product rule [$f'(x) = f'g + fg'$].

$$f'(x) = \frac{d}{dx}\left[\frac{3}{2}x^4 - 4x^3\right] = \left(\frac{3}{2}x - 4\right) \cdot \frac{d}{dx}x^3 + x^3 \cdot \frac{d}{dx}\left(\frac{3}{2}x - 4\right)$$

$$f'(x) = 6x^3 - 12x^2 = \left(\frac{3}{2}x - 4\right) \cdot [3x^2] + \frac{3}{2}x^3$$

$$f'(x) = 6x^3 - 12x^2 = \frac{9}{2}x^3 - 12x^2 + \frac{3}{2}x^3$$

Set the derivative equal to zero ($f'(x) = 0$):

$$f'(x) = 6x^3 - 12x^2 = 6x^2 \cdot (x - 2) = 0$$

The **Critical Numbers** (CN), are the following.

$$CN = \begin{cases} x = 0 \\ x = 2 \end{cases}$$

Example Problem #6 – Cont'd

Step 2:

Find the **Critical Points** of the functions [$CP = \big(x, f(x)\big)$]:

Using the Critical Numbers (CN) $\rightarrow$ ($x = 0$) and ($x = 2$), substitute into the original function ($f(x)$).

$$f(x) = x^3 \cdot \left(\frac{3}{2}x - 4\right)$$

$$f(0) = (0)^3 \cdot \left(\frac{3}{2}(0) - 4\right) = 0$$

$$f(2) = (2)^3 \cdot \left(\frac{3}{2}(2) - 4\right) = -8$$

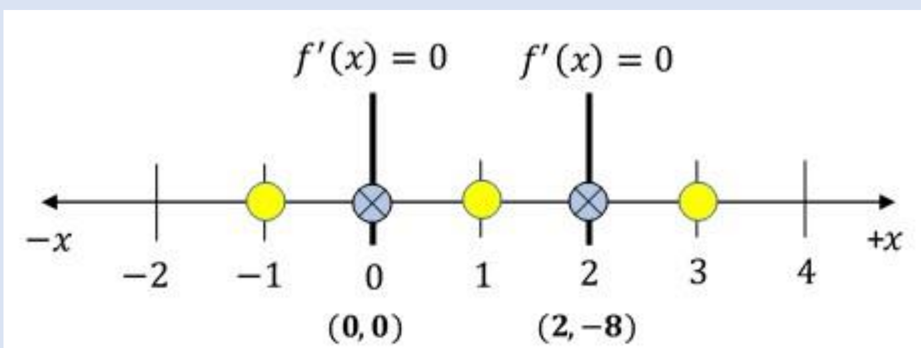
The **Critical Points** (CP), are the following.

$$CP = \begin{cases} \big(0, f(0)\big) = (0, 0) \\ \big(2, f(2)\big) = (2, -8) \end{cases}$$

Step 3:

Create a Sign Diagram or Sketch Graph, which includes Critical Numbers ($f'(x) = 0$) and arbitrary Test Points, along the **domain of the function**.

The Tests Points, are single arbitrary values, selected on the right-side and left-side of the Critical Numbers.



Example Problem #6 – Cont'd

Step 4:

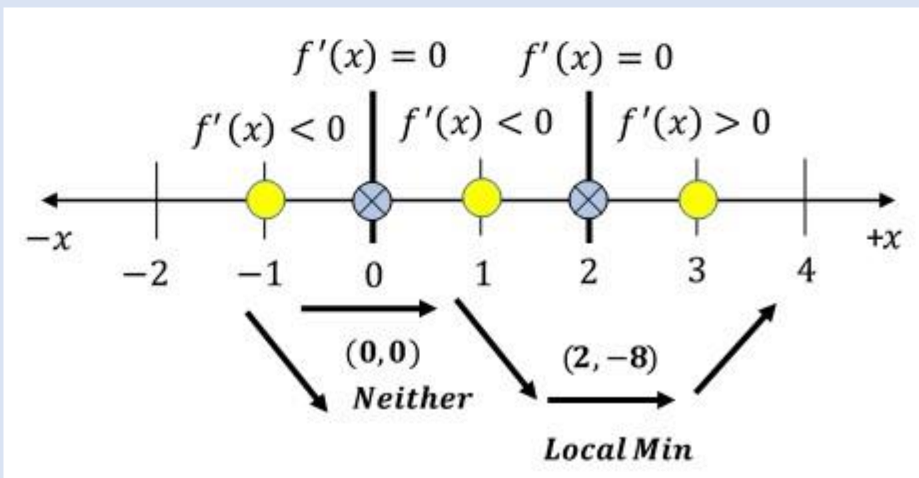
Create arbitrary Test Points, and substitute into the derivative of the function ($f'(x)$); where the **derivative is less than zero, the slope is decreasing** ($f'(x) < 0$), and where the **derivative is greater than zero, the slope is increasing** ($f'(x) > 0$):

$$f'(x) = 6x^3 - 12x^2 = 6x^2 \cdot (x - 2)$$

$$f'(-1) = (-1)^2 \cdot (-1 - 2) = -3 < 0 \rightarrow Decreasing$$

$$f'(1) = (1)^2 \cdot (1 - 2) = -1 < 0 \rightarrow Decreasing$$

$$f'(3) = (3)^2 \cdot (3 - 2) = 9 > 0 \rightarrow Increasing$$



Step 5:

Find Open Intervals of Increase (OI) & Decrease (OD), and finalize results:

Open Interval of Increase $\rightarrow OI = (2, \infty)$

Open Interval of Decrease $\rightarrow OD = (-\infty, 0)\ U\ (0, 2)$

$$Local\ Minimum = (2, -8)$$

$$Neither\ (Max\ nor\ Min) = (0, 0)$$

Example Problem #6 – Cont'd

Step 6:

Find the **Inflection Points** (IP), of the function and curve ($f(x)$), by taking the second derivative of the above function, and setting equal to zero ($f''(x) = 0$).

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}[6x^3 - 12x^2]$$

$$f''(x) = 18x^2 - 24x = 6x \cdot (3x - 4) = 0$$

The **Inflection Points** (IP), are the following.

$$IP = \begin{cases} x = 0 \\ x = \frac{4}{3} = 1.33 \end{cases}$$

Step 7:

Find the **Critical Inflection Points** of the functions [$CIP = \left(x, f(x)\right)$]:

Using the Inflection Points (IP) $\rightarrow$ ($x = 0$) and ($x = \frac{4}{3} = 1.33$), substitute into the original function ($f(x)$).

$$f(x) = x^3 \cdot \left(\frac{3}{2}x - 4\right)$$

$$f(0) = (0)^3 \cdot \left(\frac{3}{2}(0) - 4\right) = 0$$

$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 \cdot \left(\frac{3}{2}\left(\frac{4}{3}\right) - 4\right) = -\frac{128}{27} = -4.74$$

The **Critical Inflection Points** (CIP), are the following.

$$CP = \begin{cases} \left(0, f(0)\right) = (0, 0) \\ \left(1.33, f(1.33)\right) = (1.33, -4.74) \end{cases}$$

Example Problem #6 – Cont'd

Step 8:

Next perform the **Second Derivative Test** – to find the **local minimum** or **local maximum** points.

Substitute the Critical Numbers $(CN) \rightarrow (x = 0)$ and $(x = 2)$, into the second derivative equation $(f''(x))$.

$$f''(x) = 18x^2 - 24x = 6x \cdot (3x - 4) = 0$$

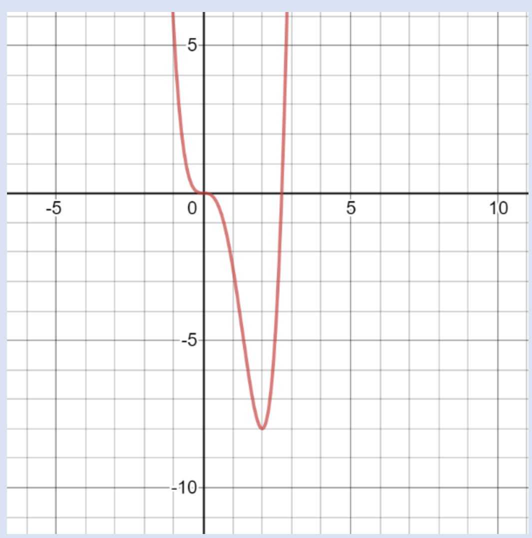
$$f''(0) = 6(0) \cdot (3(0) - 4) = 0 \rightarrow \textbf{\textit{Neither Max. or Min.}}$$

$$f''(2) = 6(2) \cdot (3(2) - 4) = 24 > 0 \rightarrow \textbf{\textit{Local Min.}}$$

The **Critical Inflection Points** (CIP), are the following.

$$CP = \begin{cases} (0, f(0)) = (0,0) \rightarrow Neither\ Max.or\ Min. \\ (2, f(2)) = (2,-8) \rightarrow Local\ Min. \end{cases}$$

Where the second derivative is greater than zero, $(f''(x) = 24 > 0)$, there is a "**Local Minimum**" and where the second derivative is zero, $(f''(x) = 0)$, the result is "**Inconclusive**", and is **neither** a local maximum nor a local minimum value.



Example Problem #6 – Cont'd

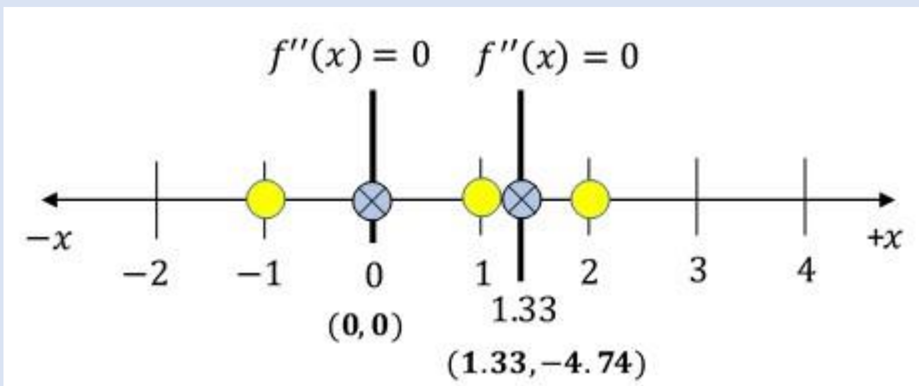
Step 9:

Test for Concavity. Create a Sign Diagram or Sketch Graph, which includes Inflection Points ($f''(x) = 0$) and arbitrary Test Points, along the **domain of the function**.

The Tests Points, are single arbitrary values, selected on the right-side and left-side of the **Inflection Points**.

The **Critical Inflection Points** (CIP), are the following.

$$CIP = \begin{cases} (0, f(0)) = (0,0) \\ (1.33, f(1.33)) = (1.33, -4.74) \end{cases}$$



Example Problem #6 – Cont'd

Step 10:

Test for Concavity. Create arbitrary Test Points, and substitute into the second derivative function ($f''(x)$).

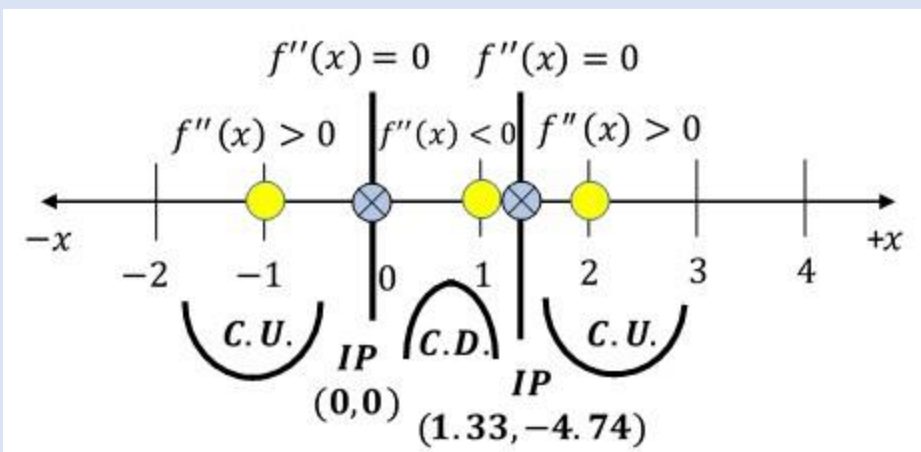
Where the **second derivative is less than zero**, the curve is "**Concave Down**" ($f''(x) < 0$), and where the **second derivative is greater than zero**, the curve is "**Concave Up**" ($f''(x) > 0$):

$$f''(x) = 18x^2 - 24x = 6x \cdot (3x - 4) = 0$$

$$f''(-1) = 6(-1) \cdot (3(-1) - 4) = 42 > 0 \rightarrow Concave\ Up$$

$$f''(1) = 6(1) \cdot (3(1) - 4) = -6 < 0 \rightarrow Concave\ Down$$

$$f''(2) = 6(2) \cdot (3(2) - 4) = 24 > 0 \rightarrow Concave\ Up$$



Step 11:

Find Open Intervals of Concavity Up (CU) & Concavity Down (CD), and finalize results:

Open Interval of Concavity Up $\rightarrow CU = (-\infty, 0), (1.33, \infty)$

Open Interval of Concavity Down $\rightarrow CD = (0, 1.33)$

Example Problem #6 – Cont'd

Conclusions:

The **Critical Points** (CP), are the following.

$$CP = \begin{cases} (0, f(0)) = (0,0) \\ (2, f(2)) = (2,-8) \end{cases}$$

$$\text{Open Interval of Increase} \rightarrow OI = (2, \infty)$$

$$\text{Open Interval of Decrease} \rightarrow OD = (-\infty, 0), (0, 2)$$

$$Local\ Minimum = (2,-8)$$

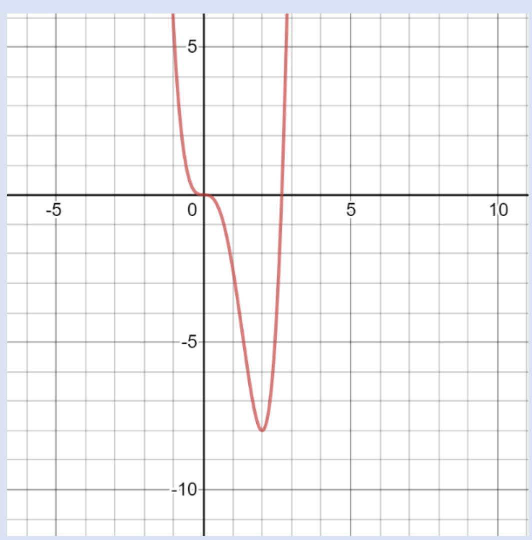
$$Neither\ (Max\ or\ Min) = (0,0)$$

The **Critical Inflection Points** (CIP), are the following.

$$CIP = \begin{cases} (0, f(0)) = (0,0) \\ \left(\frac{4}{3}, f\left(\frac{4}{3}\right)\right) = \left(\frac{4}{3}, -4.74\right) \end{cases}$$

$$\text{Open Interval of Concavity Up} \rightarrow CU = (-\infty, 0), (1.33, \infty)$$

$$\text{Open Interval of Concavity Down} \rightarrow CD = (0, 1.33)$$



2.3 - EXERCISES

Given the following functions, make a sign diagram for the first and second derivative and find all the critical points, inflection points, local mins and maximums, intervals of increase and decrease, concavity intervals and use them to sketch the graph of the function:			
1.	$f(x) = 2x^3 - 3x^2 - 36x + 28$	2.	$f(x) = (x-2)^{2/3}$
3.	$f(x) = x^4 + 4x^3 + 4x^2 - 3$	4.	$f(x) = 3x^5 - 5x^3$
5.	$f(x) = x^2(4-x)^2$	6.	$f(x) = x^3 - 3x^2 + 3x - 4$
7.	$f(x) = \sqrt[3]{x+1}$		

Sketch the graph of a continuous function f so that			
8.	$f(1) = 3$, $f\,'(1) = 0$, the point $(1,3)$ is a local maximum of f	9.	$f(2) = 1$, $f\,'(2) = 0$, the point $(2,1)$ is a local minimum of f
10.	$f(5) = 4$, $f\,'(5) = 0$, the point $(5,4)$ is not a local minimum or maximum of f .	11.	f is not differentiable at $x = 2$, $f(0) = 3$, $f'(x) > 0 \; on \; (2, \infty)$, $f'(x) < 0 \; on \; (-\infty, 2)$, f is concave down everywhere

12.	$f(0) = 3$, $f\,'(x) > 0$ on $(2,\infty)$ and $(-4, 0)$, $f'(x) < 0$ on $(-\infty,-4)$ and $(0,2)$, $f\,"(x) > 0$ on $(-\infty,-2) \cup (1,\infty)$ $f\,"(x) < 0$ on $(-2, 1)$	13.	f is undefined at x = 3 $f'(x) < 0$ on $(-\infty,3) \cup (3,\infty)$ $f\,"(x) < 0\ on\ (-\infty,3)$ $f\,"(x) > 0\ on\ (3,\infty)$

14.	In the following problems, find the critical numbers of the functions, then **use the Second Derivative Test** to determine whether the function has a local maximum, a local minimum or neither at each of those critical numbers. a) $f(x) = 2x^3 - 15x^2 + 6$, b) $g(x) = x^3 - 3x^2 - 9x + 7$, c) $h(x) = x^4 - 8x^3 + 18x^2 + 10$,

Solutions:

1. Local max (-2, 72), local min (3, - 53),
   inflection point (0.5, 9.5),
   increasing intervals (-∞, -2) and (3, ∞),
   decreasing intervals (-2, 3),
   concave up on (0.5, ∞), concave down on (-∞, 0.5).

2. Local max none, local min (2, 0),
   inflection point none,
   increasing intervals (2, ∞), decreasing intervals (-∞, 2),
   concave up never, concave down on (-∞, ∞)

3. Local max (-1, -2), local min (0, - 3) and (-2, -3),
   inflection point (-0.432, -2.56), (-1.577, -2.56),
   decreasing intervals (-∞, -2) and (-1, 0),
   increasing intervals (-2, -1) and (0, ∞),
   concave down on (-1.577, -0.423), concave up on (-∞, -1.577), (-0.423, ∞).

4. Local max (-1, 2), local min (1, -2),
   inflection points (0, 0), ($\sqrt{0.5}$, -1.237), ($-\sqrt{0.5}$,1.237),
   increasing intervals (-∞, -1) and (1, ∞), decreasing intervals (-1, 1),
   concave up on ($-\sqrt{0.5}$, 0), ($\sqrt{0.5}$, ∞), concave down on (-∞, $-\sqrt{0.5}$), (0, $\sqrt{0.5}$)

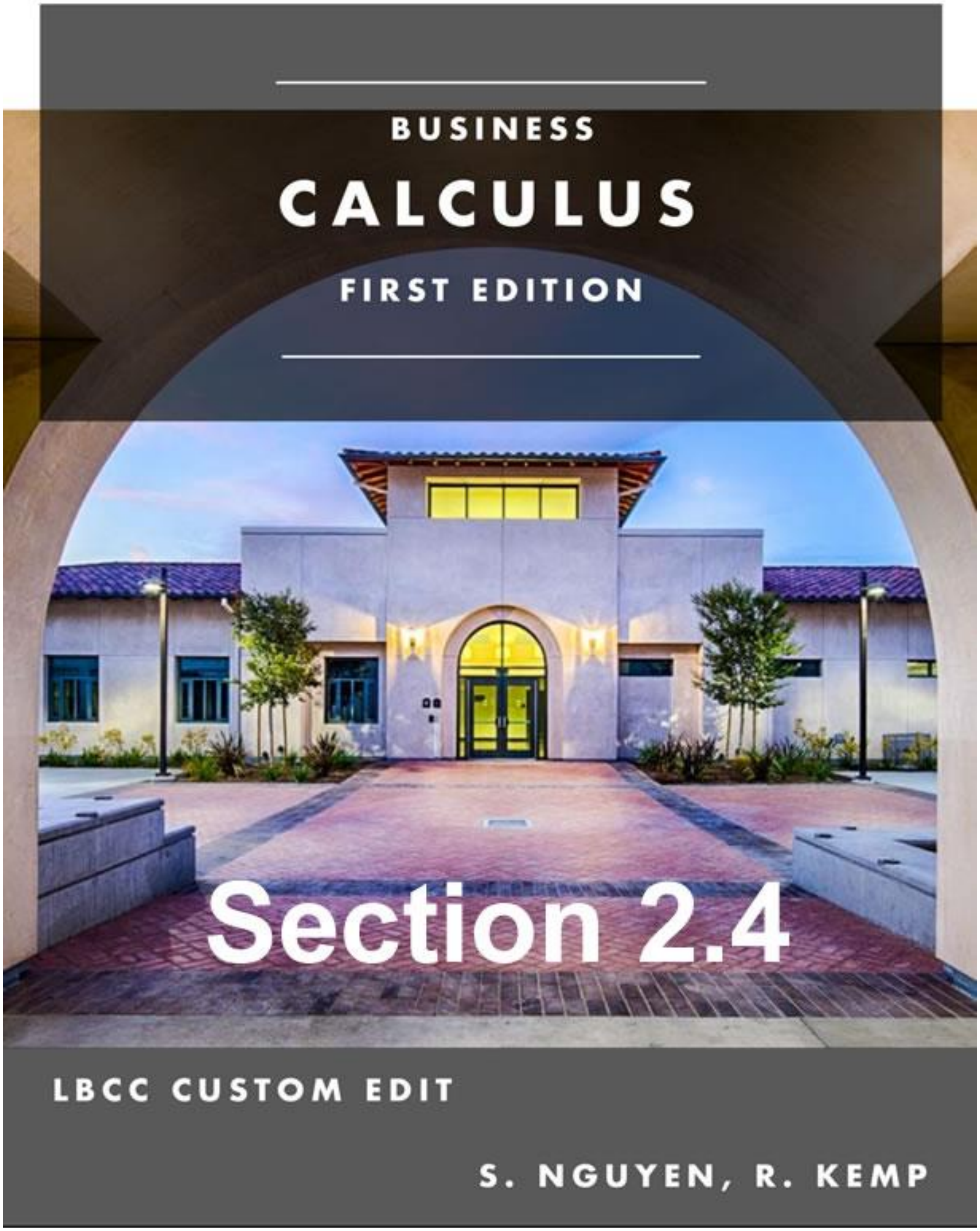
7. Local max (2, 16), local min (0, 0), (4, 0),
   increasing intervals (0, 2) and (4, ∞), decreasing intervals (−∞, 0), (2, 4),
   Inflection Points : (3.155, 7.11) & (0.845, 7.11),
   concave down on (0.845, 3.155), concave up on (-∞, 0.854), (3.155, ∞).

8. Local max none, local min none,
   inflection point (1, -3), increasing intervals (-∞, ∞),
   concave up on (1, ∞), concave down on (-∞, 1).

9. Local max none, local min none,
   inflection point (-1, 0), increasing intervals (-∞, ∞),
   concave down on (-1, ∞), concave up on (-∞, -1).

8- 13 Answers may vary.

14. a) (0, 6) is a local maximum, (5, -119) is a local minimum.
    b) (-1, 12) is a local maximum, (3, -20) is a local minimum.
    c) (0, 10) is a local minimum. (3, 37) is neither a min nor a max.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2.4 - APPLIED OPTIMIZATION

Global Maxima and Minima

In applications, we often want to find the global extreme value of a function. Just knowing that a critical point is a local extreme is not enough.

For example, if we want to make a profit, we want to make the absolutely greatest profit of all. This is called the "Global" or "Absolute" profit value.

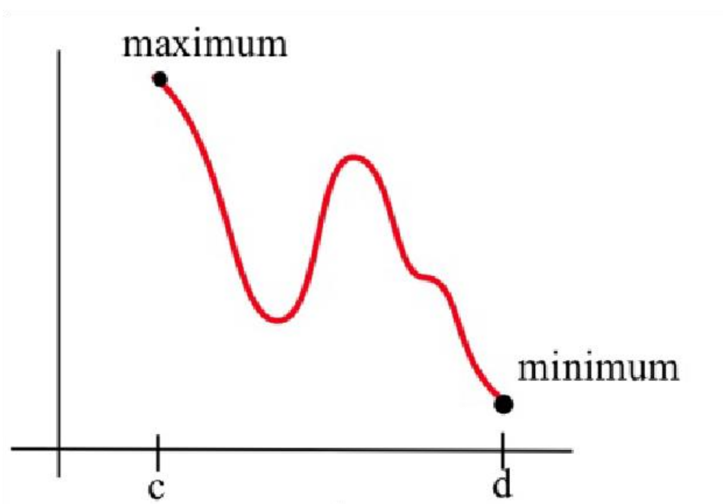
In this section we will learn how to find the global maximum and minimum values of a curve or fuction? There are just a few additional things to think about.

Endpoint Extremes

The local extremes of a function occur at critical points – these are points in the function that we can find by thinking about the shape (and using the derivative to help us).

But if we're looking at a function on a closed interval, the endpoints could be extremes.

These endpoint extremes are not related to the shape of the function; they have to do with the interval, the window through which we're viewing the function.



In the graph above, it appears that there are three critical points – one local min, one local max, and one that is neither one.

The global maximum value, the highest point of all, is at the left endpoint. The global minimum value, the lowest point of all, is at the right endpoint.

How do we decide if endpoints are global max. or min. values?

It's easier than you expected – simply plug in the endpoints, along with all the critical numbers, and compare all the y-values.

Finding a global (absolute) Minimum or Maximum value of a continuous function on a closed interval.

A continuous function on a closed interval [a, b] will have both an absolute min and max value in that interval.

(a) Find all critical numbers of the function that fit in the interval [a, b].

(b) Substitute these critical numbers and the endpoints [a, b] into the function.

- The largest ($y = f(x)$) value is the global max
- The smallest ($y = f(x)$) value is the global min.

(c) When in doubt, graph the function to be sure.

Example Problem #1

Find the global max and min of:

$$f(x) = x^3 - 3x^2 - 9x + 5 \quad \text{on the interval} \quad [-2, 6]$$

Solution:

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 - 9x + 5) = 3x^2 - 6x - 9$$

$$f'(x) = 3(x + 1)(x - 3)$$

Next set the derivative equal to zero and find the critical numbers.

$$f'(x) = 3(x + 1)(x - 3) = 0$$

The critical numbers are: ($x = -1$) and ($x = 3$).

Both these critical points are in the interval $[-2, 6]$

x	-2	-1	3	6

Let's now find the y-values for the critical points and the end poitns:

Test all of the points along the interval:

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$$

$$f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$$

$$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 5 = 3$$

$$f(6) = (6)^3 - 3(6)^2 - 9(6) + 5 = 59$$

x	-2	-1	3	6
$f(x)$	3	10	-22	59

The global minimum is (3, -22) and the global maximum is (6, 59)

If there's only one critical point

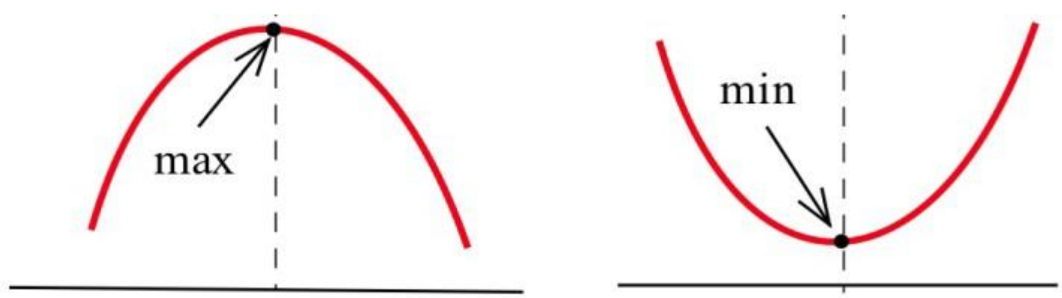
If the function has only one critical point and it's a local max (or min), then it must be the global max (or min). To see this, think about the geometry.

Look at the graph on the left – there is a local max, and the graph goes down on either side of the critical point.

Suppose there was some other point that was higher – then the graph would have to turn around.

But that turning point would have shown up as another critical point.

If there's only one critical point, then the graph can never turn back around.



We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values.

More typically, someone will describe a problem and ask your help in maximizing or minimizing something:

"What is the largest volume package which the post office will take?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?"

In this section, we'll discuss how to find these extreme values using calculus and the second derivative.

Let's recall the Second Derivative Test for Extremes, which we will use a lot in this section.

The Second Derivative Test for Extremes:

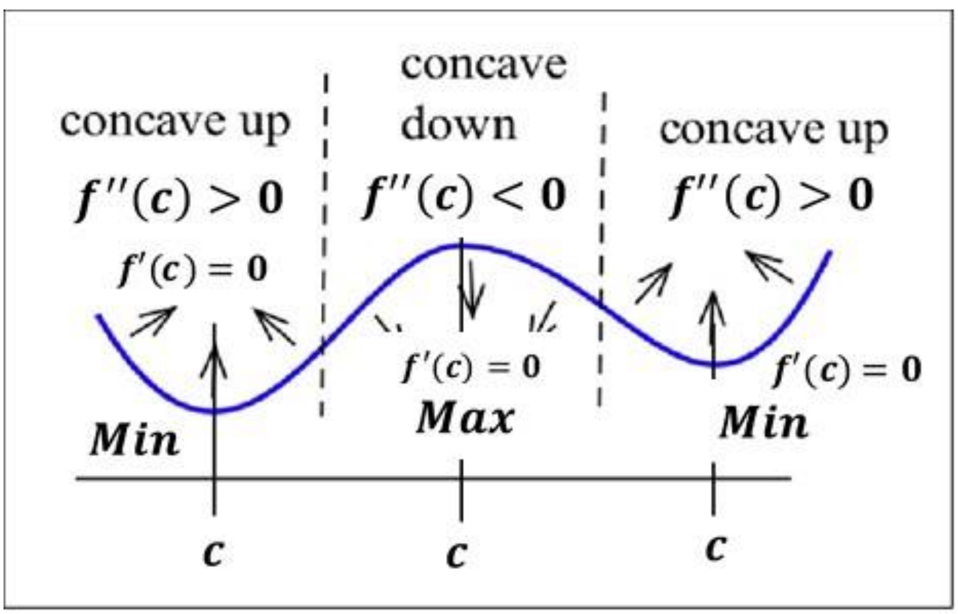
Find all critical points of (f).

For those critical points where $(f'(c) = 0)$, find $(f''(c))$.

(d) If $(f''(c) < 0)$ then (f) is **concave down** and has a **local maximum** at $(x = c)$.

(e) If $(f''(c) > 0)$ then (f) is **concave up** and has a **local minimum** at $(x = c)$.

(f) If $(f''(c) = 0)$ then (f) may have a **local maximum**, **local minimum**, or **neither** at $(x = c)$.



Max/Min Applications

Let's say we need to solve the following problem:

The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment.

Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot.

The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.

The process of finding maxima or minima is called optimization.

The function we're optimizing is called the **objective function**. The objective function can be recognized by its proximity to "est" words (greatest, least, highest, farthest, most...).

Look at the garden store example; the cost function is the objective function. In many cases, there are two (or more) variables in the problem.

In the garden store example, again, the length and width of the enclosure are both unknown.

If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable.

Equations that relate the variables in this way are called **constraint equations**. The constraint equations are always equations, so they will have equals signs.

For the garden store, the fixed area relates the length and width of the enclosure. This will give us our constraint equation.

Max-Min Story Problem Technique:

(a) Translate the English statement of the problem line by line into a picture (if that applies) and into math. This is often the hardest step!

(b) Identify the objective function. Look for words indicating a largest or smallest value.

(c) If you seem to have two or more variables, find the constraint equation. Think about the English meaning of the word “constraint,” and remember that the constraint equation will have an equals sign.

(d) Solve the constraint equation for one variable and substitute into the objective function. Now you have an equation of one variable.

(e) Use calculus to find the optimum values. (Take derivative, find critical points, test. Don’t forget to check the endpoints!)

(f) Look back at the question to make sure you answered what was asked. Translate your number answer back into English.

Example Problem #2

The manager of a garden store wants to build 600 sq. ft. rectangular enclosure on the store's parking lot. Three sides of the enclosure will be built of redwood fencing, which costs \$7/foot and the fourth side will be built of cement block which cost \$14/foot. Find the dimensions of the least costly such enclosure.

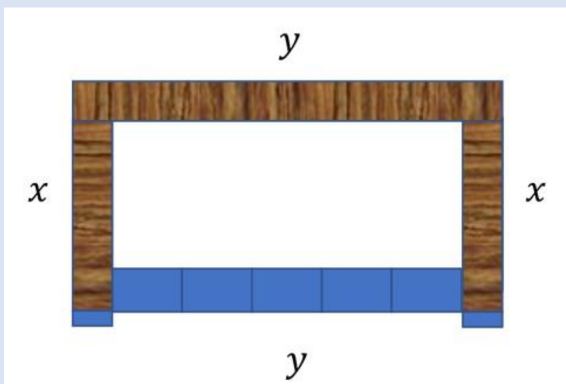
Solution:

First let's translate the problem:

"The manager of a garden store wants to build 600 sq. ft. rectangular enclosure on the store's parking lot"

Let x and y be the dimensions of the rectangle, with y being the side made of blocks.

Then the area of the rectangle is A = xy = 600



"Three sides of the enclosure will be built of redwood fencing, which costs \$7/foot and the fourth side will be built of cement block which cost \$14/foot."

The cost of the fencing will then be:

$$C = 7x + 7y + 7x + 14y = 14x + 21y$$

"Find the dimensions of the least costly such enclosure."

Find (x) and y), such that (C) is minimized.

The objective function is the cost function, and we want to minimize it.

Example Problem #2 – Cont’d

As it stands, though, it has two variables, so we need to use the constraint equation. The constraint equation is the fixed area (A):

$$A = xy = 600$$

Solve this equation for y to get ($y = \frac{600}{x}$), and then substitute into (C):

$$C(x) = 14x + 21\left(\frac{600}{x}\right) = 14x + 12600x^{-1}$$

Now we have a function of only one variable, so we can minimize it.

Find the derivative of (C) and set it equal to (0) to find the critical points.

$$C'(x) = 14 - 12600x^{-2} = 0$$

$$14 = \frac{12600}{x^2} \rightarrow 14x^2 = 12600 \rightarrow x^2 = 900 \rightarrow x = \pm 30$$

Since (x) is a side of a rectangle the negative answer does not make sense, therefore

($x = 30$) and substituting back ($y = \frac{600}{30} = 20$)

To test if this (x -value) produces a minimum we need to run the second derivative test:

Since:

$$C''(x) = 25200x^{-3} \quad , \quad C''(30) > 0$$

So, at ($x = 30$) we have a **local minimum**.

Since this is the only critical point in the domain, this must be the global minimum. Going back to our constraint function, we can find that when *(*$y = 20$*)*, *(*$x = 30$*)*.

The dimensions of the enclosure that minimize the cost are 20 feet by 30 feet.

Example Problem #3

Maximizing the Volume of a Box:

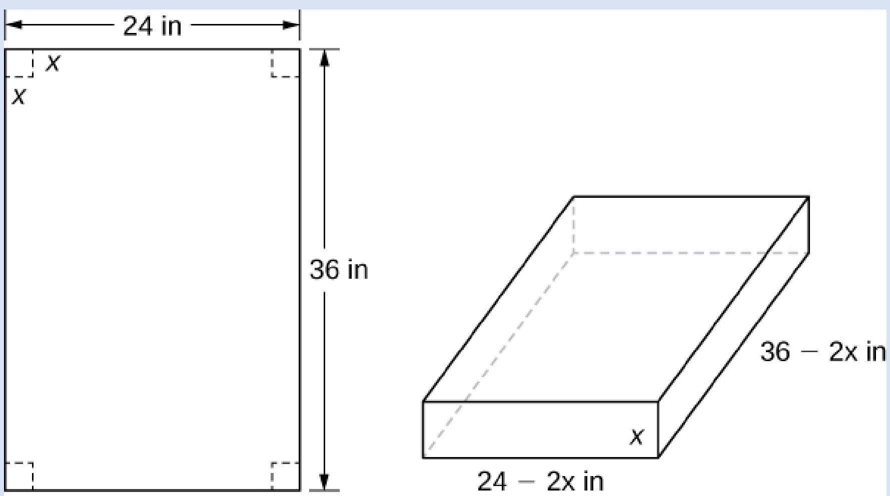
An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side.

What size square should be cut out of each corner to get a box with the maximum volume?

Solution:

Step 1: Let x be the side length of the square to be removed from each corner Then, the remaining four flaps can be folded up to form an open-top box.

Let (V) be the volume of the resulting box.



Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize (V).

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is $(V = L \cdot W \cdot H)$, where L, W, and H are the length, width, and height, respectively.

Step 4: From the picture, we see that the height of the box is x inches, the length is 36 − 2x inches, and the width is 24 − 2x inches.

Therefore, the volume $(V(x))$ of the box is:

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x$$

Example Problem #3 – Cont'd

The volume ($V(x)$) of the box is:

$$V(x) = 4x^3 - 120x^2 + 864x$$

Step 5: To determine the domain of consideration, let's examine the picture. Certainly, we need (x > 0).

Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24in.; otherwise, one of the flaps would be completely cut off.

Therefore, we are trying to determine whether there is a maximum volume of the box for (x) over the open interval (0, 12).

Since (V) is a continuous function over the closed interval [0, 12], we know (V) will have an absolute maximum over the closed interval.

Therefore, we consider (V) over the closed interval [0, 12] and check whether the absolute maximum occurs at an interior point.

Step 6: Since ($V(x)$) is a continuous function over the closed interval [0, 12], (V) must have an absolute maximum (and an absolute minimum).

Since V(x) = 0 at the endpoints and ($V(x) > 0$) for (0 < x < 12), the maximum must occur at a critical point.

The derivative is:

$$V'(x) = 12x^2 - 240x + 864$$

To find the critical points, set the derivative, equal to zero, ($V'(x) = 0$), and solve the equation

$$V'(x) = 12x^2 - 240x + 864 = 0$$

Dividing both sides of this equation by (12), the problem simplifies to solving the equation.

$$V'(x) = x^2 - 20x + 72 = 0$$

Next, use the quadratic formula to solve for (x) in the above equation.

Example Problem #3 – Cont'd

Using the quadratic formula, we find that the critical points are

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(72)}}{2}$$

$$x = 10 \pm 2\sqrt{7}$$

Since ($x = 10 + 2\sqrt{7}$) is not in the domain of consideration, the only critical point we need to consider is ($x = 10 - 2\sqrt{7}$).

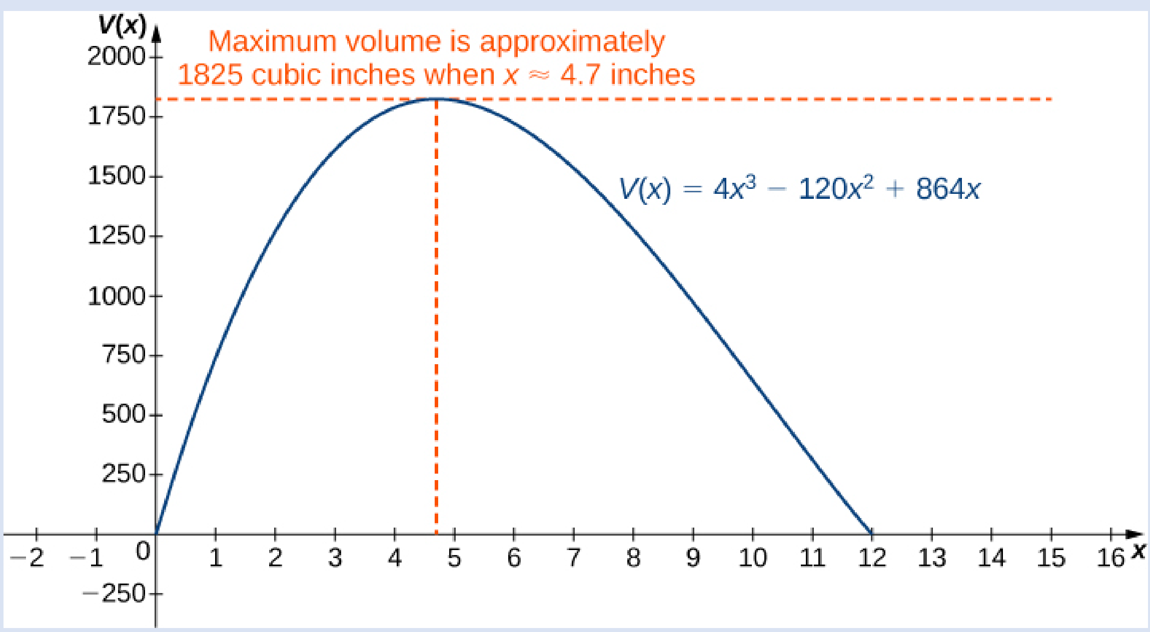
Therefore, the **volume is maximized** if we let ($x = 10 - 2\sqrt{7}\ in$).

The maximum volume is:

$$V(x) = 4x^3 - 120x^2 + 864x$$

$$V\left(10 - 2\sqrt{7}\right) = 640 + 448\sqrt{7} \approx 1825\ in^3$$

as shown in the following graph.



"Marginal Revenue = Marginal Cost"

You may have heard before that "profit is maximized when marginal cost and marginal revenue are equal." Let's explore what that statement means in the mathematics below.

Suppose we want to maximize profit. We find the profit function, then find its critical points, test them, etc.

But remember that Profit = Revenue – Cost.

$$P(x) = R(x) - C(x)$$

So, Profit' = Revenue' – Cost'. That is, the derivative of the profit function is

$$MP(x) = P'(x) = \frac{d}{dx}P(x) = \frac{d}{dx}R(x) - \frac{d}{dx}C(x)$$

$$MP(x) = MR(x) - MC(x)$$

Now let's find the critical points – those will be where the Profit is equal to zero or undefined:

$$MP(x) = P'(x) = 0 \quad or\ Undefined$$

And, when,

$$MP(x) = P'(x) = MR(x) - MC(x) = 0$$

Therefore,

$$MR(x) = MC(x)$$

Profit has critical points when Marginal Revenue and Marginal Cost are equal.

In all the cases we'll see in this class, "Profit" will be very well behaved, and we won't have to worry about looking for critical points where Profit' is undefined.

But remember that not all critical points are local max or local minimum values!

There are places where ($MR(x) = MC(x)$) could represent local max, local min, or neither one.

Example Problem #4

A company sells *q* ribbon winders per year at \$*p* per ribbon winder. The price function for ribbon winders is given by:

$$p = 300 - 0.02q$$

In other words, the price will be adjusted depending on the quantity of ribbon winders you are planning to buy.

The ribbon winders cost \$30 apiece to manufacture, plus there are fixed costs of \$9000 per year. Find the quantity where profit is maximized.

Solution:

We want to maximize profit, but there isn't a formula for profit given. So, let's make one.

The Price Function: $p(q) = 300 - 0.02q$

We can find a function for Revenue ($R(q)$) using the price function for ($p(q)$).

$$R(q) = p(q)q$$

$$R(q) = (300 - 0.02q)q = 300q - 0.02q^2$$

We can also find a function for Cost, using the variable cost of \$30 per ribbon winder, plus the fixed cost:

$$C(q) = 9000 + 30q$$

Putting them together, we get a function for Profit:

$$P(q) = R(q) - C(q)$$

$$P(q) = 300q - 0.02q^2 - 9000 - 30q$$

$$P(q) = -0.02q^2 + 270q - 9000$$

Example Problem #4 – Cont'd

Now we have two choices.

We can find the critical points of Profit by taking the derivative of *P*(*q*) directly, or we can find MR and MC and set them equal.

(Naturally, you'll get the same answer either way.)

The derivative of the Profit is given by,

$$MP(q) = \frac{d}{dq}P(q) = \frac{d}{dq}(-0.02q^2 + 270q - 9000)$$

$$MP(q) = P'(q) = -0.04q + 270$$

$$MP(q) = -0.04q + 270 = 0$$

$$q = \frac{270}{0.04} = 6750$$

Next, let's use $(MR(q) = MC(q))$ this time.

$$MR(q) = \frac{d}{dq}(300q - 0.02q^2) = 300 - 0.04p$$

$$MC(q) = \frac{d}{dq}(9000 + 30q) = 30$$

$$MR(q) = MC(q) = 300 - 0.04p = 30$$

$$q = \frac{270}{0.04} = 6750$$

The only critical point is at $(q = 6750)$. Now we need to be sure this is a local max and not a local min.

Example Problem #4 – Cont'd

To check for maximization, take the second derivative of the profit function.

$$P''(q) = \frac{d}{dq}P'(q) = \frac{d}{dq}(-0.04q + 270) = -0.04$$

$$P''(q) = -0.04 < 0$$

Therefore, the profit function is maximized.

And since it's the only critical point, it must also be the global max.

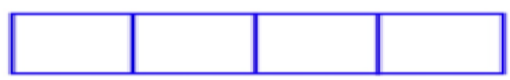
Profit is maximized when they sell 6750 ribbon winders.

2.4 - EXERCISES

Find the absolute minimums and maximums of the following functions on the given intervals

1.	$f(x) = x^2 - 6x + 5$ on $[-2, 5]$	2.	$f(x) = 2 - x^3$ on $[-2, 1]$
3.	$f(x) = x^3 - 3x + 5$ on $[-2, 1]$	4.	$f(x) = \frac{x}{x^2 + 1}$ on $[-4, 4]$

5. You have 120 feet of fencing to construct a pen with 4 equal sized stalls. If the pen is rectangular and shaped like the one below, what are the dimensions of the pen of largest area and what is that area?

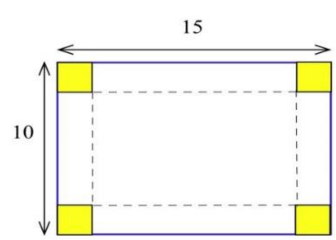


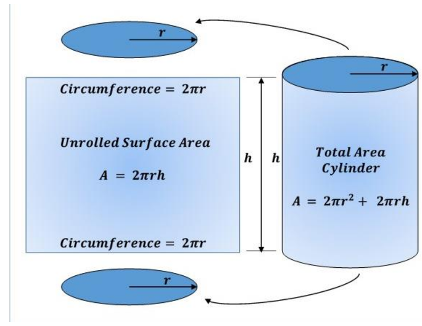
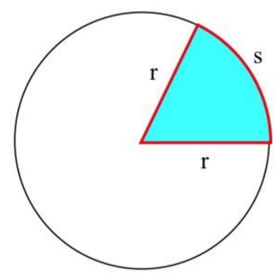
6. Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides.

What are the dimensions of the new garden of largest area?

7. You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides.

How much should you cut from each corner so the resulting box has the greatest volume?



8.	Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, cost 5¢ and cost 3¢ per square inch, respectively for the top, bottom and sides.  <u>Hint:</u> (Constraint) - Volume of Cylinder – $V = \pi r^2 h$ (Objective) - Total Cost of Cylinder $C_{Total} = C_{Top} + C_{Bottom} + C_{Side}$ $C_{Total} = \pi r^2 \cdot \left(\frac{cost}{in^2}\right) + \pi r^2 \cdot \left(\frac{cost}{in^2}\right) + 2\pi rh \cdot \left(\frac{cost}{in^2}\right)$
9.	You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" shown. The area of such a sector is $\left(A = \frac{1}{2}(r \cdot s)\right)$.  What value of r maximizes the enclosed area? <u>Hint:</u> The fencing sides are described by the outsides of the shaded circular area.
10.	A company makes and sells toys. The total cost in dollars for the company to produce an amount *q* of a cetain toy is given by $C(q) = 135 + 2q + 0.15q^2$ Find the quantity that minimizes the **average cos**t for producing that toy. Also find the average cost and the total cost for that quantity.

11.	. The total cost in dollars for Alicia to make q oven mitts is given by . $$C(q) = 64 + 1.5q + 0.01q^2$$ (a) What is the fixed cost? (b) Find a function that gives the marginal cost. (c) Find a function that gives the average cost. (d) Find the quantity that minimizes the average cost. (e) Confirm that the average cost and marginal cost are equal at your answer to part (d).
12.	Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where 50 ≤ p ≤ 200, the number of cars n they rent per day can be modeled by the linear function $$n(p) = 1000 - 5p.$$ If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?
13.	Consider a pizzeria that sell pizza for a revenue of $R(x) = 10x$ and a cost of $C(x) = 2x + x^2$ dollars. How many pizzas sold maximizes the profit?
14.	Go Fly Motors, produces a four-motor airplane. It costs \$500 to produce each single aircraft motor and the fixed cost associated with producing the motors is \$200 per day. The price function is: $$p(x) = 800 - 7.5x$$ Where (p) is the price in dollars at which (x) aircraft motors will be sold. Find: a) The cost function of producing the motors. b) The profit function from producing and selling the motors. c) The quantity of motors that should be produced, and the price at which they should be sold in order to maximize the profit. d) The maximum Profit.

Solutions:

1. Absolute minimum is (3, -4), absolute maximum is (-2, 21)

2. Absolute minimum is (1, 1), absolute maximum is (-2, 10)

3. Absolute minimum is (-2, 3) and (1, 3), absolute maximum is (-1, 7)

4. Absolute minimum is (-1, - 0.5), absolute maximum is (1, 0.5)

5. 30ft by 12 ft; $A''(12) = -5$; therefore, Maximized.

6. 40ft by 40ft; $A''(40) = -2$; therefore, Maximized.

7. Cut about 1.96 in. , Volume = 132.03 ft^3, $V''(1.96) = -52.96$; Therefore Maximized

8. about 2.389 in (radius) and 5.575 in (height)

9. S = 50 ft , r = 25ft, A = 625ft^2 , $A''(25) = -2$; Therefore Maximized

10. 30 toys at an average cost of \$11/toy and a total cost of \$330.
$AC''(30) = 0.01$; Therefore Minimized

11. a) \$64, b) $MC(q) = 1.5 + 0.02q$, c) $AC(q) = 64q^{-1} + 1.5 + 0.01q$
d) 80 oven mitts, $MC(80) = AC(80) = \$3.1$/ oven mitt.

12. Charge \$100/day for a total revenue of \$50,000.

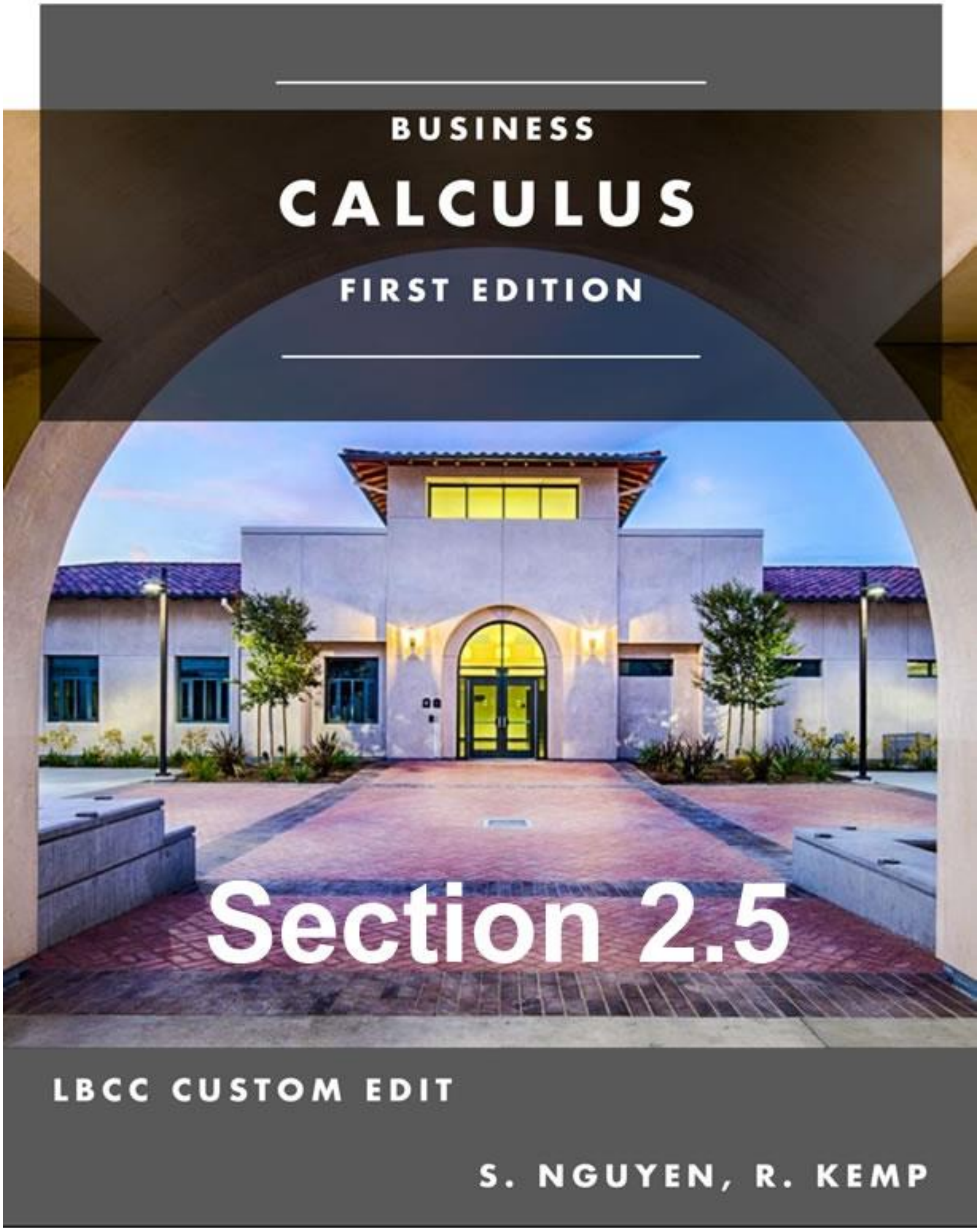
13. 4 pizzas,
Profit = $P(4) = \$16$, Revenue = $R(4) = \$40$, Cost = $C(4) = \$\,24$
$P(4)'' = -2$; Therefore Maximimized

14. a) $C(x) = 500x + 2500$

b) $P(x) = x(800 - 7.5x) - (500x + 200)$

c) Produce 20 aircraft motors and sell for $p(20) = \$650$ per aircraft motor,

d) Maximum profit of $P(20) = \$2{,}800$, $P''(20) = -15$;Therefore Maximized.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2.5 - MORE APPLICATIONS - OPTIMIZATION

Price raises/ decreases in increments

When trying to maximize their revenue, businesses also face the constraint of consumer demand. While a business would love to sell lots of products at a very high price, typically demand decreases as the price of goods increases and demand increases if the prices are being lowered. We will revisit this idea many times this semester.

In simple cases, we can construct that demand curve to allow us to maximize revenue. In this section, we will discuss some applications in which the demand increases or decreases in chunks based on the price being decreased or increased in chunks.

Example Problem #1

A concert promoter has found that if she sells tickets for $50 each, she can sell 1200 tickets, but for each $5 she raises the price, 50 less people attend.

What price should she sell the tickets at to maximize her revenue?

Solution:

Method 1:

We are trying to **maximize revenue**, and we know that the revenue (R) is the product, of the **price per ticket** (p), multiplied by the, **quantity of tickets** (q) sold.

Revenue: $R = pq$

The problem provides information about the demand relationship between price and quantity - as price increases, demand decreases.

We need to find a formula for this relationship. To investigate, let's calculate what will happen to attendance if we raise the price:

Price, p	50	55	60	65
Quantity, q	1200	1150	1100	1050

Example Problem #1 – Cont'd

You might recognize this as a linear relationship.

We can find the slope for the relationship by using two points:

$$m = \frac{\Delta q}{\Delta p} = \frac{1150 - 1200}{55 - 50} = -10$$

You may notice that the second step in that calculation corresponds directly to the statement of the problem: the attendance drops 50 people for every $5 the price increases.

Using the point-slope form of the line, we can write the equation relating price and quantity:

$$q - 1200 = -10(p - 50)$$

Simplifying to slope-intercept form gives the quantity (q) of demand equation:

$$q = 1700 - 10p$$

Substituting this into our revenue equation, we get an equation for revenue involving only one variable:

$$R = pq = p(1700 - 10p) = 1700p - 10p^2$$

Now, we can find the maximum of this function by taking the derivative (R'), and finding critical numbers.

$$R' = \frac{d}{dp}(1700p - 10p^2)$$

$$R' = 1700 - 20p$$

$$R' = 1700 - 20p = 0$$

$$p = \frac{1700}{20} = 85$$

Example Problem #1 – Cont'd

Therefore, the price is ($p = \$85$) per ticket.

Using the second derivative test, ($R'' = -20 < 0$), so the critical number is a **local maximum**.

$$R'' = \frac{d}{dp}(R') = \frac{d}{dp}(1700 - 20p) = -20$$

Since it's the only critical number, we can also conclude it's the global maximum.

The promoter will be able to maximize revenue by charging ($p = \$85$) per ticket.

At this price, the quantity (q) she will sell:

$$q = 1700 - 10\,p$$

$$q = 1700 - 10\,(85) = 850\,Tickets$$

The revenue is therefore:

$$R = pq = (\$85)(850) = \$72{,}250$$

generating $72,250 in revenue.

Example Problem #1 – Cont'd

Method 2:

Let x = number of $5 price raises

Revenue again equals to price/item times numbers of items.

$$R(x) = p(x)q(x)$$

We are starting with $50/ ticket and selling 1200 tickets.

The price rises by $5 for every (x), and the number of tickets decreases by 50 for every (x).

$$p(x) = 50 + 5x$$

$$q(x) = 1200 - 50x$$

$$R(x) = (50 + 5x)(1200 - 50x) = 60000 + 3500x - 250x^2$$

Taking the derivative,

$$R'(x) = \frac{d}{dx}(60000 + 3500x - 250x^2)$$

$$R'(x) = 3500 - 500x$$

Setting it equal to $(R'(x) = 0)$ to find the critical numbers, means (x = 7).

$$R'(x) = 3500 - 500x = 0$$

$$x = \frac{3500}{500} = 7$$

Using the second derivative test, $(R'' = -500 < 0)$, so the critical number is a local maximum.

$$R''(x) = \frac{d}{dx}R'(x) = \frac{d}{dx}(3500 - 500x) = -500$$

Example Problem #1 – Cont'd

Since it's the only critical number, we can also conclude it's the global maximum.

Price:

$$p(x) = 50 + 5x$$

$$p(7) = 50 + 5(7) = 85$$

$$p(x) = \$85 \ per\ ticket$$

Quantity:

$$q(x) = 1200 - 50x$$

$$q(7) = 1200 - 50(7) = 850$$

$$q(x) = 850 \ Tickets$$

Revenue again equals to price/item times numbers of items.

$$R(x) = p(x)q(x)$$

$$R(x) = p(x)q(x) = (\$85)(850) = \$72{,}250$$

So, the price to charge ($p(x) = \$85 \ per\ ticket$) and the quantity of tickets sold is ($q(x) = 850 \ Tickets$), will be sold, for a total amount of revenue equal to ($R(x) = \$72{,}250$).

Example Problem #2

A farmer decided to go into business growing apple trees. The farmer estimates that if she plants 95 apple trees per acre, each apple tree will yield 74 baskets of apples.

She estimates that for each additional apple tree, that she will plant per acre, the yield for each apple tree, will decrease by two (2) baskets.

How many apple trees should she plant per acre to maximize harvest?

Solution:

We want to maximize the harvest yield. Therefore, we let (x) equal the number of added apple trees per acre.

$$x = \textit{Number of additional apple trees per acre}$$

Next, let

Apples Trees per acre – ($T(x)$)

$$T(x) = 95 + x$$

Yield per Apple Tree (Baskets of Apples) – ($A(x)$)

$$A(x) = 74 - 2x$$

Next, we can calculate the "Total Yield" ($Y(x)$) from the apple trees.

$$Y(x) = T(x) \cdot A(x)$$

$$Y(x) = (95 + x) \cdot (74 - 2x) = -2x^2 - 116x + 7030$$

Next, we maximize the Total Yield ($Y(x)$), by taking the derivative of the Total Yield ($Y'(x) = 0$) and setting equal to zero.

$$Y'(x) = \frac{d}{dx}(-2x^2 - 116x + 7030) = -4x - 116 = 0$$

Example Problem #2 – Cont'd

Next solving for (x).

$$Y'(x) = -4x - 116 = 0$$

$$x = Number\ of\ additional\ apple\ trees\ per\ acre$$

$$x = -\frac{116}{4} = -29\ \ Apple\ Trees$$

Next, we will test for maximization by using the second derivative test.

$$Y''(x) = \frac{d}{dx}Y'(x) = \frac{d}{dx}(-4x - 116) = -4$$

$$Y''(x) = -4 < 0 \ :\ Therefore\ Maximized$$

The number of additional apple trees equal to ($-29\ \ Apple\ Trees$); this means that the farmer should plant 29 less/fewer apple trees per acre.

Apples Trees per acre – ($T(x)$)

$$T(x) = 95 + x = 95 - 29 = 66$$

$$T(x) = 66\ \ Apple\ Trees\ per\ acre$$

Yield per Apple Tree (Baskets of Apples) – ($A(x)$)

$$A(x) = 74 - 2x = 74 - 2(-29) = 74 + 58 = 132$$

$$A(x) = 132\ \ Baskets\ of\ Apples\ per\ tree$$

Minimizing Inventory Costs

Suppose that a toy store has found out that it sells about 1000 dolls of a certain type in a year. Should the store order all 1000 dolls on the first day of the year?

The store would not want to do such a thing, as storage alone would cost about $3/doll for the year.

Also, the store would not want to order the dolls too often, as ordering costs alone for shipping would be $15 per order, plus of course, there if a $5 charge/doll from the manufacturer.

The store needs to find a balance and would like to find out how many dolls to order at one time and how many times per year to order, in order to **minimize inventory costs**.

Let's call (x = lot size), i.e. the number of dolls the store needs to order at one time.

Since the store needs to have ordered 1000 dolls per year, as this is how many it sells in a year, there will be ($\frac{1000}{x}$) many orders per year.

We do have two types of costs:

Storage costs pertain to the costs of storing the dolls in one year.

Since the store orders (x) dolls at one time and starts selling them, and then orders again (x) dolls when the number of dolls has pretty much dropped down to (0), we can say that we have on average ($\frac{x}{2}$) many dolls in the store at any time.

Since it costs $3 to store one doll per year, the storage costs will equal:

$$(\#dolls\ in\ the\ store)\left(\frac{storage\$}{doll}\right) = \frac{x}{2}\cdot 3 = 1.5x$$

Ordering costs pertain to the costs of ordering the dolls.

Each time an order is made, there is a \$200 fixed charge, plus there is a \$5 charge/doll:

$$\left(\# \frac{orders}{year}\right)\left(\frac{cost}{order}\right) = \left(\frac{1000}{x}\right)(5x + 15) = 5000 + \frac{15,000}{x}$$

The cost function that we need to minimize is:

$$C(x) = 1.5x + 15,000x^{-1} + 5000$$

$$C'(x) = 1.5 - 15,000x^{-2}$$

To find the critical points, we need to set ($C'(x) = 0$) and solve:

$$C'(x) = 1.5 - 15,000x^{-2} = 0$$

$$1.5 = \frac{15,000}{x^2}$$

$$x^2 = \frac{15,000}{1.5} = 10,000$$

$$x = \pm 100$$

Since we cannot order a negative amount of dolls, only the positive 100 answer makes sense.

So, we need to order 100 dolls at one time ($\frac{1000}{100} = 10$) times per year.

Minimizing production runs

Now, let's look from the point of the manufacturer. Let's say a toy factory knows that about 1000 of certain doll will be ordered in one year. Should the factory produce all 1000 dolls on the first day of the year?

That would not be a good idea, as it would cost the factory \$2 per doll to store for the year.

On the other hand, every time the factory sets up the machines to produce this doll, it would cost \$160 in fixed set-up costs plus \$2 to manufacture every doll.

How many times per year and how many dolls at one time should the factory produce to minimize costs?

We do have two types of costs:

Storage costs pertain to the costs of storing the dolls in one year.

Since the store produces (x) dolls at one time and starts sending them to stores, then produces (x) dolls again when the number of dolls has pretty much dropped down to 0, we can say that they have on average $(\frac{x}{2})$ many dolls in the storage at any time.

Since it costs \$2 to store one doll per year, the storage costs will equal

$$(\#dolls\ in\ the\ store)\left(\frac{storage\$}{doll}\right) = \frac{x}{2}\cdot 2 = 1x$$

Manufacturing costs pertain to the costs of manufacturing the dolls.

Each time a production run is set up, there is a \$160 fixed set up cost, plus there is a \$2 cost/doll to manufacture.

$$\left(\#\frac{runs}{year}\right)\left(\frac{cost}{run}\right) = \left(\frac{1000}{x}\right)(2x + 160) = 2000 + \frac{160{,}000}{x}$$

The cost function that we need to minimize is:

$$C(x) = 1x + 160{,}000x^{-1} + 2000$$

$$C'(x) = 1 - 160{,}000x^{-2}$$

To find the critical points, we need to set ($C'(x) = 0$) and solve:

$$C'(x) = 1 - 160{,}000x^{-2} = 0$$

$$1 = \frac{160{,}000}{x^2}$$

$$x^2 = \frac{160{,}000}{1} = 160{,}000$$

$$x = \pm 400$$

Since we cannot produce a negative amount of dolls, only the positive 400 answer makes sense.

So, we need to produce 400 dolls at one time ($\frac{1000}{400} = 2.5$) times per year, i.e. five times in the span of two years.

2.5 - EXERCISES

1.	A store sells 2500 packages containing a dozen paper towel rolls per year. It costs \$10 per year to store one such package. There is a fixed \$20 cost every time the store reorders these packages and there is \$9 cost per package. How many packages of these paper towels at one time and how many times per year should the store reorder to minimize inventory costs?
2.	A factory produces about 720 skateboards in one year. It would cost the factory \$20 per skateboard to store for the year. On the other hand, the set-up production costs are \$200 each time, plus \$15 to manufacture every skateboard. How many times per year should the factory manufacture the skateboards and how many at one time to minimize production costs?
3.	A store sells about 100 ping-pong tables per year. It costs \$20 to store one table per year. Reordering costs are \$40/order plus \$100 per table. How many times per year should the store reorder and how many tables at one time in order to minimize inventory costs? Also, find the inventory costs.
4.	A factory produces about 360 WIFI routers in one year. It would cost the factory \$8 per router to store for the year. On the other hand, the set-up production costs are \$10 plus \$8 to manufacture every router. How many times per year should the factory manufacture the routers and how many at one time to minimize production costs? Also, find the production costs.
5.	A retail store sells about 600 printers per year. It costs \$16 to store each printer per year. Reordering costs are \$300 per order and it costs the store \$40 per printer. How many times per year should the store reorder and how many printers at one time in order to minimize inventory costs? Also, find the inventory costs.

6.	A fitness club charges a $50 membership price per month. At this price, the club can attract 504 members. For every 2 dollars decrease in membership fees, the club can attract 24 more members. What membership price would maximize **revenue** and how many members will sign up? What is the maximum revenue?
7.	You have a garden row of 20 watermelon plants that produce an average of 30 watermelons a piece. For any additional watermelon plant planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant and how many watermelons will they each produce in order to maximize the **yield**?
8.	When a movie theater charges $12 for admission, there is an average attendance of 240 people per day. For every $1.50 price increase, there is a loss of 15 customers per day. What admission should be charged to maximize **revenue**? How many people will be attending at this price?
9.	A rental car company can rent 20 cars per day at a price of $34 per day. For every $5 daily price increase, two less cars can be rented each day. If each rented car costs the company $4 in maintenance and fixed costs are $40 per day, how many cars should be rented and what rental price should be charged to maximize the daily **profit**? What is the maximum profit?
10.	You own a small airplane which holds a maximum of 20 passengers. It costs you $100 per flight from St. Thomas to St. Croix for gas and wages, plus an additional $6 per passenger for the extra gas required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by $1 per passenger for each passenger over 10 who flies (that is, if you fly 11 passengers, they each pay $29, if 12 fly they each pay $28...). What number of passengers on a flight will maximize your **profits** and what will be the profit?

Solutions:

1. Order 100 packages at one time, 25 times per year.

2. Produce 120 skateboards at one time, 6 times per year.

3. Order 20 ping-pong tables at one time, 5 times per year for a cost of $10,400.

4. Produce 30 routers at one time, 12 times per year, for a total cost of $3,120.

5. Order 150 printers at one time, 4 times per year, for a total cost of $26, 400.

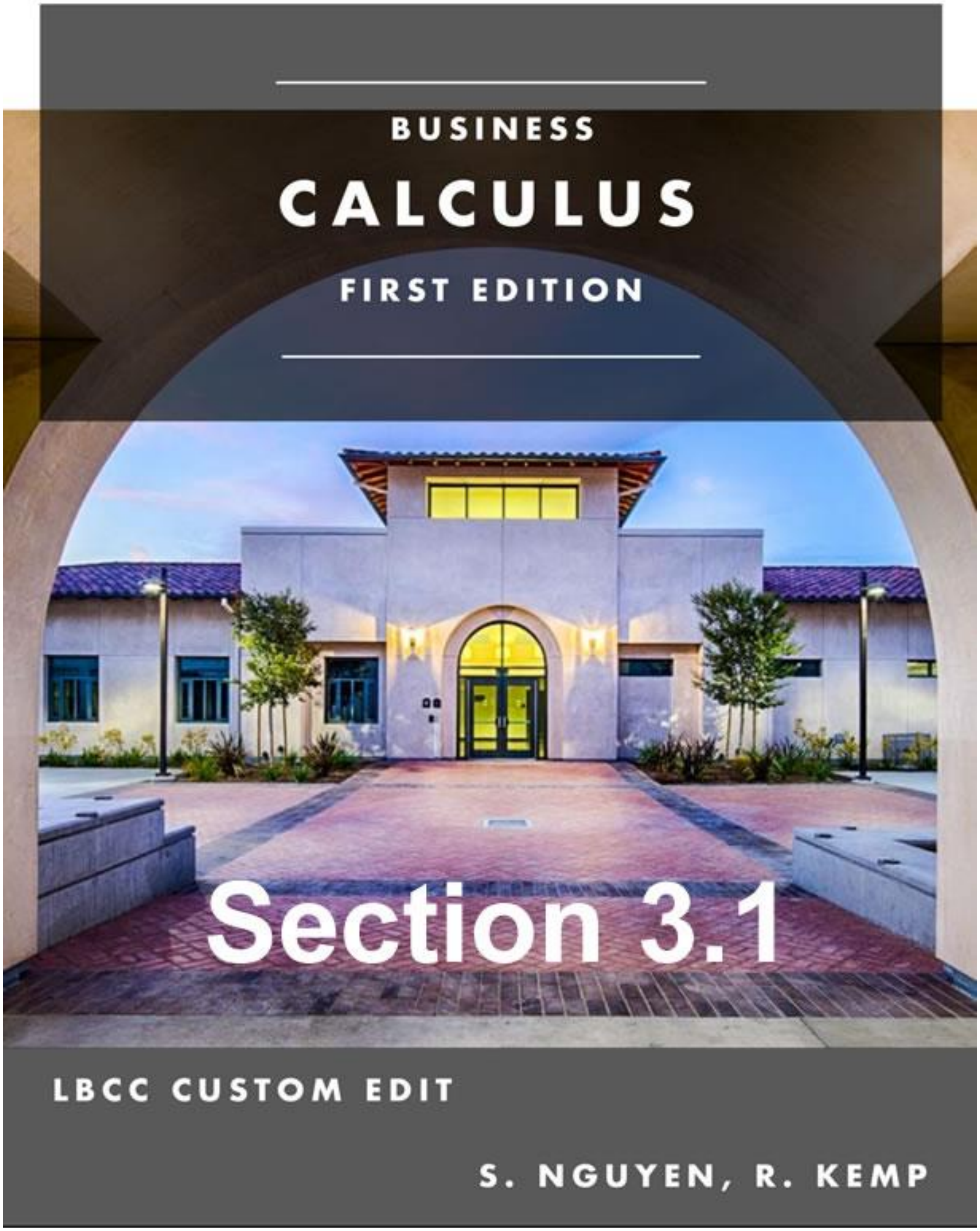
6. Charge $46 per membership, then there will be 552 members,
for a maximum monthly revenue of $25,392
Revenue is Maximized - $R"(x) = -96$

7. Plant 25 watermelon plants which will each produce 25 watermelons.
Yield is Maximized - $Y''(x) = -2$

8. Charge $18 per ticket and 180 people will be attending.
Revenue is Maximized - $R"(x) = -45$

9. Rent 16 cars at $44 each per day, for a maximum daily profit of $600.
Profit is Maximized - $P''(x) = -20$

10. Fly 17 passengers at $23/passenger, for a total profit of $189.
Profit is Maximized - $P''(x) = -2$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3.1 - EXPONENTIAL FUNCTIONS

Consider these two companies:

Company *A* has 100 stores, and expands by opening 50 new stores a year

Company *B* has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Company *A* is exhibiting linear growth. In linear growth, we have a constant rate of change – a constant *number* that the output increased for each increase in input.

For company A, the number of new stores per year is the same each year.

Company *B* is different – we have a *percent* rate of change rather than a constant number of stores/year as our rate of change.

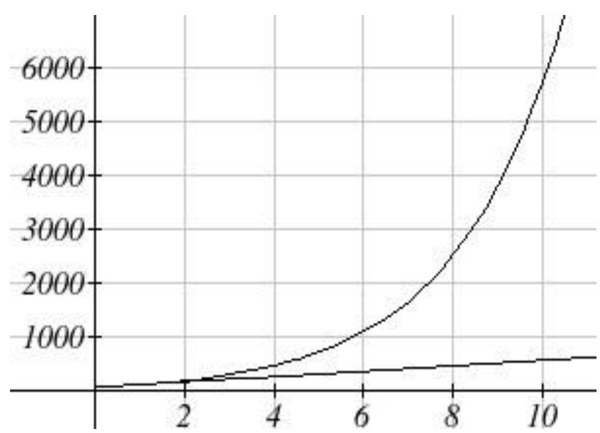
To see the significance of this difference, compare a 50% increase when there are 100 stores to a 50% increase when there are 1000 stores:

100 stores, a 50% increase is 50 stores in that year.

1000 stores, a 50% increase is 500 stores in that year.

Calculating the number of stores after several years, we can clearly see the difference in results.

Years (t)	**Company *A*** $f(t) = 100 + 50t$	**Company *B*** $f(t) = 100(1 + 0.5)^t$
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



This percent growth can be modeled with an exponential function.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form:

$$y = f(x) = a \cdot (1 + r)^x = a \cdot b^x$$

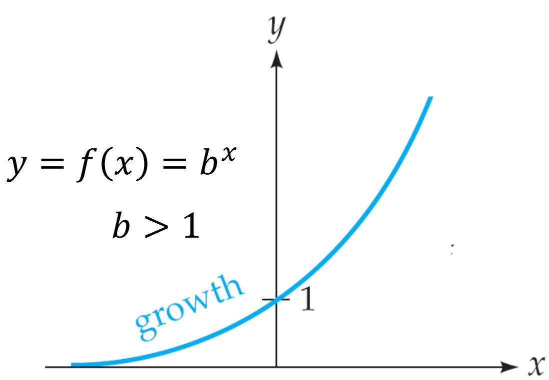
$$where, \quad b = 1 + r$$

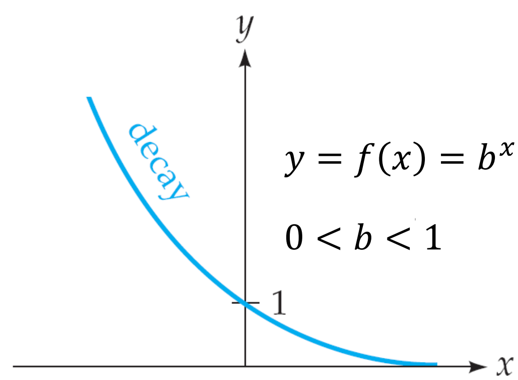
- Initial Starting value – a
- Percent Growth/Decay Rate – r
- Period of Time – x

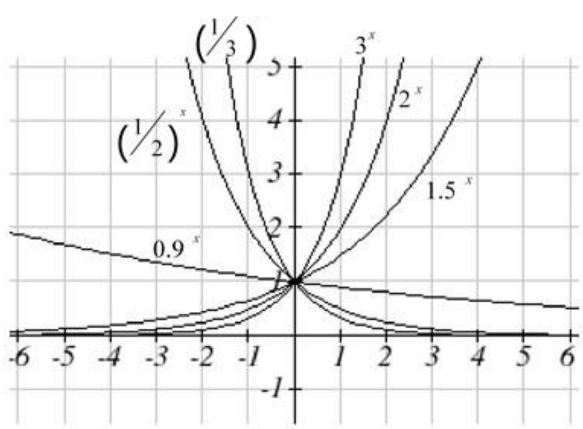
Where (a) is, the initial or starting value of the function, and (r) is, the percent growth or decay rate, written as a decimal, and the Period of Time (x).

And, the exponentential function base (b) is, the growth factor or growth multiplier.

Since powers of negative numbers behave strangely, we limit *b* to positive values.







Example Problem #1

India's population was 1.14 billion in the year 2008 and is growing by about 1.34% each year. Write an exponential function for India's population, and use it to predict the population in 2020.

Solution:

Using 2008 as our starting time (t = 0), our initial population will be 1.14 billion.

Since the percent growth rate was 1.34%, our value for (r) is ($r = \frac{1.34}{100} = 0.0134$).

a) Next, using the basic formula for exponential growth

$$y = f(t) = a \cdot (1 + r)^t$$

- Initial Population value – $a = 1.14 \ \ billion\ people$
- Rate – $r = 0.0134$
- Time – $t = (2020 - 2008) = 12\ years$

$$y = f(t) = 1.14 \cdot (1 + 0.0134)^t = 1.14 \cdot (1.0134)^t$$

b) To estimate the population in 2020, we evaluate the function at ($t = 12$), since 2020 is 12 years after 2008.

$$y = f(12) = 1.14 \cdot (1 + 0.0134)^{12} = 1.14 \cdot (1.0134)^{12}$$

$$y = f(12) = 1.337 \ \ billion\ people\ in\ 2020$$

Periodic Compounding Formulas

Compound Interest

If an amount of money (P), called **principal or present value**, is invested at an **interest rate** r, for a period of **time** (t), measured in years, and the money is **compounded** n times per year, meaning the interest is added to the principal (n) times per year, the amount of money (A) accrued, or the value after t years, will be

$$P(t) = P\left(1+\frac{r}{n}\right)^{n\cdot t}$$

$where,$

- Principle value – P
- Rate in Percent – r
- Compounding Periods per year – n
- Time in # of Years – t

Periodic Compounding (Future Value) can be calculated using the formula when the compounding period (n) per year, multiplied by the time (t) is "**Positive:**

$(n \cdot t > 0)$

$$Future\ Value \quad \rightarrow \quad P(t) = P\left(1+\frac{r}{n}\right)^{n\cdot t}$$

Periodic Compounding (Present Value) can be calculated using the formula when the compounding period (n) per year, multiplied by the time (t) is "**Negative":**

$(n \cdot t < 0)$

$$Present\ Value \quad \rightarrow \quad P(t) = \frac{P}{\left(1+\frac{r}{n}\right)^{n\cdot t}} = P\left(1+\frac{r}{n}\right)^{-n\cdot t}$$

Example Problem #2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested.

If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment grow to over those 24 months?

Solution:

Method 1:

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly.

To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: [1.2%/12 = 0.1%].

Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of *r* = 0.001, and our input *m* measured in months.

$$f(m) = 1000\left(1+\frac{0.012}{12}\right)^m = 1000(1+0.001)^m$$

After 24 months, the account will have grown to:

$$f(24) = 1000(1 + 0.001)^{24} = \$1024.28$$

Method 2:

Using the "Future Value" compound interest formula:

- Principle value – $P = \$1000$
- Rate in Percent – $r = 0.012$
- Periods per year – $n = 12\ months$
- Time in # of Years – $t = 2\ years$

$$P(t) = P\left(1+\frac{r}{n}\right)^{n\cdot t} = 1000\left(1+\frac{0.012}{12}\right)^{(12)\cdot(2)} = \$1024.28$$

Example Problem #3

A certain person came into an inheritance, and wants to lend some money. They want to raise $15,000, over seven (7) years, and are willing to charge 21% interest, compounded quarterly. What is the present value of the $15,000?

Solution:

Using the "Present Value" compound interest formula:

- Principle value – $P = \$15000$
- Rate in Percent – $r = 0.21$
- Periods per year – $n = 4$
- Time in # of Years – $t = 7\ years$

$$P(t) = \frac{P}{\left(1+\frac{r}{n}\right)^{n\cdot t}} = \frac{15000}{\left(1 + \frac{0.21}{4}\right)^{(4)\cdot(7)}} = \$3{,}579.91$$

Therefore, the present value to loan today is $3,579.91; and is the amount that the inheritor would have to loan today (in the present), to receive $15,000 at the end of seven (7) years.

Depreciating Assets and Decay Formula

Most, if not all objects and assets (cars, boats, office equipment, computers, cell phones, i.e.) "depreciate", "lose value", or "Decay" over time.

Objects and assets lose value or depreciate by a fixed percentage each year.

The measure of the depreciation of an asset, is to use the "Future Value" Compound Interest formula ($P(t) = P\left(1+\frac{r}{n}\right)^{n\cdot t}$), but with a negative interest rate ($r < 0$). And the compounding period is annually ($n = 1$).

Depreciating Asset "Future Value" Compound Interest Formula:

$$P(t) = P(1 - r)^t$$

Example Problem #4

A new car buyer select a brand-new Mercedes Benz, worth \$65,000, depreciates in value by about 38% per year. How much will the car worth after 4 years of depreciation?

Solution:

Using the Depreciating Asset "Future Value" Compound Interest Formula:

- Principle value – $P = \$65000$
- Rate in Percent – $r = 0.38$
- Periods per year – $n = 1$
- Time in # of Years – $t = 4\ years$

$$P(t) = P(1 - r)^t = 65{,}000(1 - 0.38)^4 = \$9{,}604.62$$

Therefore, after 4 years of depreciating in value, the brand-new \$65,000 Mercedes Benz, is now worth \$9,604.62.

Example Problem #5

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day.

If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution:

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate.

Our initial quantity is ($Q(0) = a = 100\ mg$), and our growth rate (r) will be negative 13%, since we are decreasing ($r = -0.13$).

a) Next, use the basic formula for exponential growth/decay

$$Q(t) = a \cdot (1 + r)^t$$

- Initial Population value – $a = 100\ mg\ of\ Bismuth - 210$
- Rate of Decay – $r = -0.13$
- Time in Years – $t = 7\ days \cdot \left(\frac{1\ year}{365\ days}\right) \approx 0.0192\ years$

$$Q(t) = 100 \cdot (1 - 0.13)^t = 100 \cdot (0.87)^t$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

b) After one week, ($t = 7\ days$), the quantity remaining would be:

$$Q(0.00192) = 100 \cdot (1 - 0.13)^{0.0192} = 100 \cdot (0.87)^{0.0192}$$

$$Q(t) \approx 99.73\ mg\ \ of\ Bismuth - 210$$

Therefore, after seven days of radioactive decay, the “Quantity” of 99.73 mg of Bismuth-210 remains.

Example Problem #6

The function ($T(q)$) represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store measured quarterly since January 1, 2010.

Interpret all the parts of the equation:

$$T(2) = 86(1.64)^2 = 231.3056$$

Solution:

Let's consider the Exponential Growth/Decay function:

$$f(t) = a \cdot (1 + r)^t$$

a) Interpreting this from the basic exponential form, we know that 86 is our initial value ($a = 86$).

This means that on Jan. 1, 2010, this region had 86,000 Android smart phone contracts.

b) From the above exponential function, we know that every quarter the number of smart phone contracts grows by ($r = 64\%$).

$$(1 + r) = 1.64 \qquad \textit{thus the rate} \quad r = 0.64 \rightarrow 64\%$$

c) The time-period ($t = 2$) means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

$$T(2) = a \cdot (1 + r)^2 = 231.3056$$

Continuous Compounding Formulas

Let's look at the compound interest formula again. What happens as n → ∞?

To answer this question, we let m = n/r and write:

$$A = P\left(1 + \frac{r}{n}\right)^{n \cdot t} = P\left(1 + \frac{1}{m}\right)^{mr \cdot t}$$

and examine the behavior of [$\left(1 + \frac{1}{m}\right)^{m}$] as m → ∞, using a table of values.

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^{m}$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $\left(1 + \frac{1}{m}\right)^{m}$ is approaching a number between 2.7 and 2.8 as m → ∞.

In fact, $\left(1 + \frac{1}{m}\right)^{m}$ does approach some number as m → ∞.

We call this number *e*. To six decimal places of accuracy,

e ≈ 2.718282.

The letter *e* was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between *e* and logarithmic functions.

We still use the notation *e* today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, **compounded continuously**, (meaning infinitely many times per year), then;

$$P(t) = P \cdot e^{rt}$$ **(continuous compound formula)**

This function may be familiar. Since functions involving base e arise often in applications, we call the function **f(x) = e^x the natural exponential function**.

Because *e* is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of *e*, usually labeled e^x.

Some computer software instead defines a function ***exp(x)***, where ***exp(x) = e^x*.**

Since calculus studies continuous change, we will almost always use the *e*-based form of exponential equations in this course.

Continuous Growth/Decay Formula

Continuous Growth/Decay can be calculated using the formula:

$$P(t) \;=\; P \cdot \mathrm{e}^{rt}$$

$where,$

- Principle value – P
- Rate in Percent – r
- Time in # of Years – t

Where the initial value is (P) *a* is the starting amount, and the rate is (r) is a continuous growth/decay rate.

Continuous Compounding (Future Value) can be calculated using the formula when the rate (r) multiplied by the time (t) is "**Positive"** ($r \cdot t > 0$)

$$Future\ Value \quad \to \quad P(t) \;=\; P \cdot \mathrm{e}^{rt}$$

Continuous Compounding (Present Value) can be calculated using the formula when the rate (r) multiplied by the time (t) is "**Negative"** ($r \cdot t < 0$)

$$Present\ Value \quad \to \quad P(t) \;=\; \frac{P}{\mathrm{e}^{rt}} \;=\; P \cdot \mathrm{e}^{-rt}$$

The concept of Continuous Compounding is different from Periodic Compounding. For example, money experiencing monthly compounding, the interest is not accrued, and will only be paid out, at the end of each month. However, with continuous compounding the interest is added to the principle amount, as soon as it is earned, and is instantly added to the principle.

Example Problem #7

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon222 decay to in 3 days?

Solution:

Since we are given a continuous decay rate, we use the continuous growth formula. Also, the substance is decaying, therefore the growth rate will be negative: ($r = -0.173$).

$$P(t) = P \cdot \mathrm{e}^{-rt}$$

- Principle value – $P = 100\ mg$
- Rate in Percent – $r = -0.173$
- Time in # of Years – $t = 3\ days \cdot \left(\frac{1\ year}{365\ days}\right) \approx 0.00822\ years$

$$P(0.00822) = 100 \cdot \mathrm{e}^{-0.173(0.00822)} \approx 99.86\ mg\ of\ Radon - 222$$

Therefore, after three days of radioactive decay, ($P(t) = 99.86\ mg$), of Radon-222 remains.

Graphs of Exponential Functions

Graphical Features of Exponential Functions

Graphically, in the function $[f(x) = a \cdot b^x]$

a is the vertical intercept of the graph *b* determines the rate at which the graph grows

the function will increase if $b > 1$

the function will decrease if $0 < b < 1$

The graph will have a horizontal asymptote at $y = 0$

The graph will be concave up if $a > 0$; concave down if $a < 0$.

The domain of the function is all real numbers

The range of the function is $(0,\infty)$

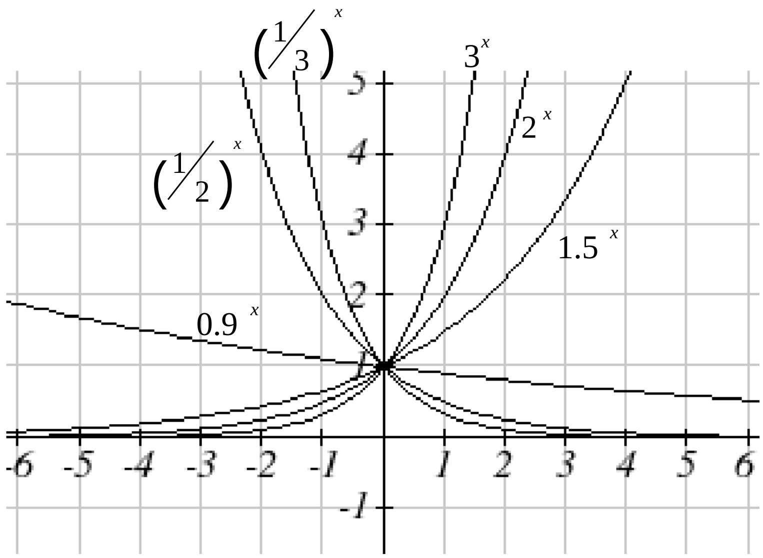
When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points (0, *a*) and (1, *ab*).

The value *b* will determine the function's long run behavior:

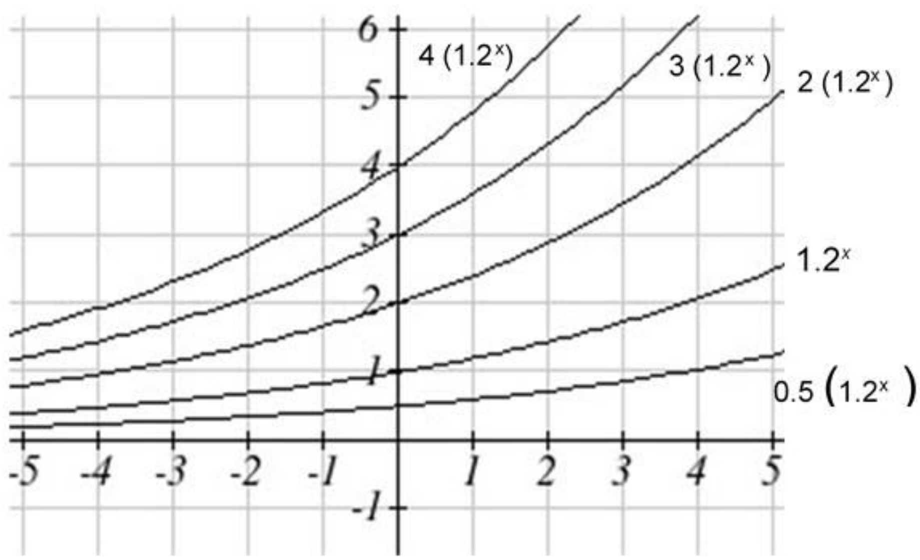
If $b > 1$, as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to 0$.

If $0 < b < 1$, as $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to \infty$

The first set shows various graphs, where (*a)* remains the same and we only change the value for (*b)*. Notice that the closer the value of (*b)* is to 1, the less steep the graph will be.



In the next set of graphs, (*a)* is altered and our value for (*b)* remains the same.



Notice that changing the value for *a* changes the vertical intercept. Since (*a)* is multiplying the (b^x) term, *a* acts as a vertical stretch factor, not as a shift.

Notice also that the long run behavior for these functions is the same because the growth factor did not change and none of these *a*- values introduced a vertical flip.

Example Problem #8

Suppose \$500 is invested in an account at an annual interest rate of r = 5.5%, compounded continuously.

Let (t) denote the number of years after the initial investment and ($P(t)$) denote the amount of money in the account at time (t).

a) Find a formula for ($P(t)$) and use it to find the amount of money in the account after 10 years and after 20 years.

b) How much money would need to be invested now in order to obtain \$1,000 two years from now, i.e. find the present value.

Solution:

a) If P dollars are invested in an account at an annual interest rate (r), compounded continuously (Future Value), then

$$P(t) = P \cdot \mathrm{e}^{rt}$$

- Principle value – $P = \$500$
- Rate in Percent – $r = 0.055$
- Time in # of Years – $t = 10\ years\ \ \&\ 20\ years$

$$P(t) = 500 \cdot \mathrm{e}^{0.055t}$$

After 10 years, the amount of money in the account is:

$$P(10) = 500 \cdot \mathrm{e}^{0.055(10)} = 500 \cdot \mathrm{e}^{0.55} = \$866.63$$

After 20 years, the amount of money in the account is:

$$P(20) = 500 \cdot \mathrm{e}^{0.055(20)} = 500 \cdot \mathrm{e}^{1.1} = \$1{,}502.08$$

Example Problem #8 – Cont'd

b) We need to find the **Continuous Compounding Present Value**:

$$P(t) = P \cdot \mathrm{e}^{-rt}$$

- Principle value – $P = \$1000$
- Rate in Percent – $r = 0.055$
- Time in # of Years – $t = 2\ years$

$$P(t) = 1000 \cdot \mathrm{e}^{-0.055t}$$

After 2 years, the Present Value of the account is:

$$P(2) = 1000 \cdot \mathrm{e}^{-0.055(2)} = 1000 \cdot \mathrm{e}^{-0.11} = \$895.83$$

The Present Value of the account undergoing continuous compounding interest, after two years is ($P(t) = \$895.83$).

Example Problem #9

Comparing Interest Rates.

The Good Money Bank can offer you two different types of Compound Interest rates of return, on your principle investment, of any amount. The bank offers two investment vehicles, one offer is 10.5% compounded continuously, and the second offer is 10.4% compounded monthly.

Which bank offer, yields a better return on your principle investment?

Solution:

The principle amount of investment is not given. Thus, any amount of principle will work, let use choose our principle investment to be one dollar ($P = \$1.00$).

Using the “Future Value” **continuous compound interest** formula:

$$P(t) = P \cdot e^{rt}$$

- Principle value – $P = \$1.00$
- Rate in Percent – $r = 0.105$
- Time in # of Years – $t = 1\ year$

$$P(t) = 1 \cdot e^{0.105(1)} = e^{0.105} = \$1.1107$$

Using the “Future Value” **periodic compound interest** formula:

$$P(t) = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$$

- Principle value – $P = \$1.00$
- Rate in Percent – $r = 0.104$
- Periods per year – $n = 12$
- Time in # of Years – $t = 1\ years$

$$P(t) = 1 \cdot \left(1 + \frac{0.104}{12}\right)^{(12)\cdot(1)} = (1.008)^{12} = \$1.1091$$

Thus, the bank offer which yields the better return on your investment is the Continuous Compound Interest result. The difference is small, but for larger principle investments, and longer periods of time, the difference would be magnified.

3.1 - EXERCISES

1.	A population numbers 11,000 organisms initially and grows by 8.5% each year. Write an exponential model for the population and use to find the number of organisms after 10 years.
2.	A vehicle purchased for \$32,500 depreciates at a constant rate of 5% each year. Determine the approximate value of the vehicle 12 years after purchase.
3.	If \$4,000 is invested in a bank account at an interest rate of 7% per year, find the amount in the bank after 9 years if interest is compounded: A) annually, B) quarterly, C) monthly, D) continuously.
4.	Find the amount of money your parents would need to invest at the time of your birth in a trust fund paying 6% interest annually, compounded continuously, so that you would get \$100,000 by the time you turn 18.
5.	According to the World Bank, at the end of 2013 (t = 0) the U.S. population was 316 million and was increasing continuously at a rate of r = 0.74% per year. Write an exponential model for the population. Based on this model, what will be the population of the United States in 2025?
6.	The demand D (in millions of barrels) for oil in an oil-rich country is given by the function D(p) = 150 $\cdot(2.7)^{-.025p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between \$15 and \$20.

7.	What would be a better investment, a rate of 9.5% compounded continuously, or a rate of 10% compounded annually? (Hint: start with the same principal and see what it grows to.)
8.	A bank advertises that it compounds money quarterly and that it will double your money in 10 years. What is the interest rate?
9.	Peter Minuit purchased Manhattan Island from the Indians in 1626 for $24 worth of merchandise. Assuming a continuous inflation rate of 6%, what would be the value of Manhattan today, in 2019?
10.	The cost of a postage stamp was 4 cents in 1962. It is now, in 2019, 55 cents. Assuming an annual exponential growth, find the growth rate, and then use to predict the cost of stamps by 2050.
11.	In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where *t* represents the time in years after 1968. a. Find a formula for $w(t)$. b. If the minimum wage was $5.15 in 1996, how does this amount compare to what the model predicts?

12.

Match the following function with the graphs.

$a)\ f\ (x) =\ 2\ (0.69)^x$

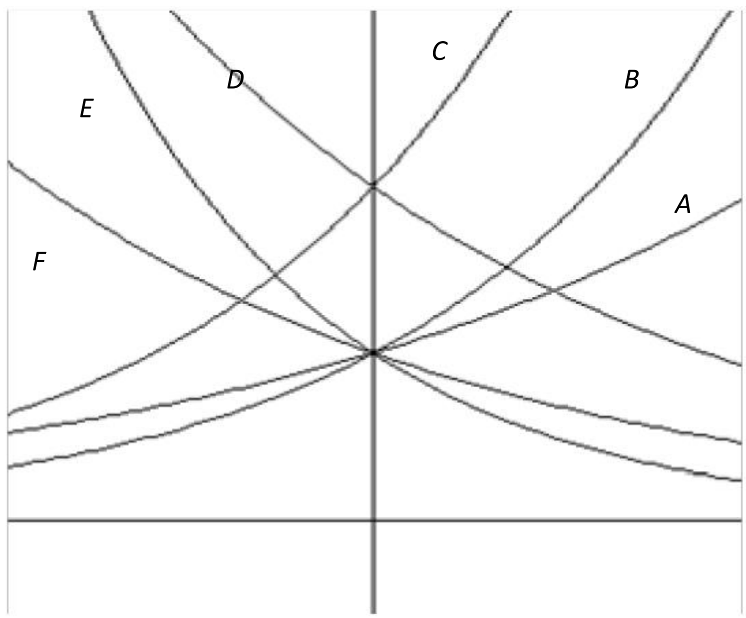
$b)\ f\ (x) =\ 2(1.28)^x$

$c)\ f\ (x\ ) =\ 2\ (0.81)^x$

$d)\ f\ (x\ ) =\ 4\ (1.28)^x$

$e)\ f\ (x\ ) =\ 2\ (1.59)^x$

$f)\ f\ (x\ ) =\ 4\ (0.69)^x$



Solutions:

1. $A(x) = 11000(1.085)^x, A(10) = 24870.82$ bacteria

2. $17,561.70

3. $7353.84, $7469.63, $7496.71, $7510.44

4. $33,959.55

5. $A(x) = 316\, e^{.0074x}$, 345 million people

6. between 103.35 and 91.29 millions of barrels of oil

7. 10%

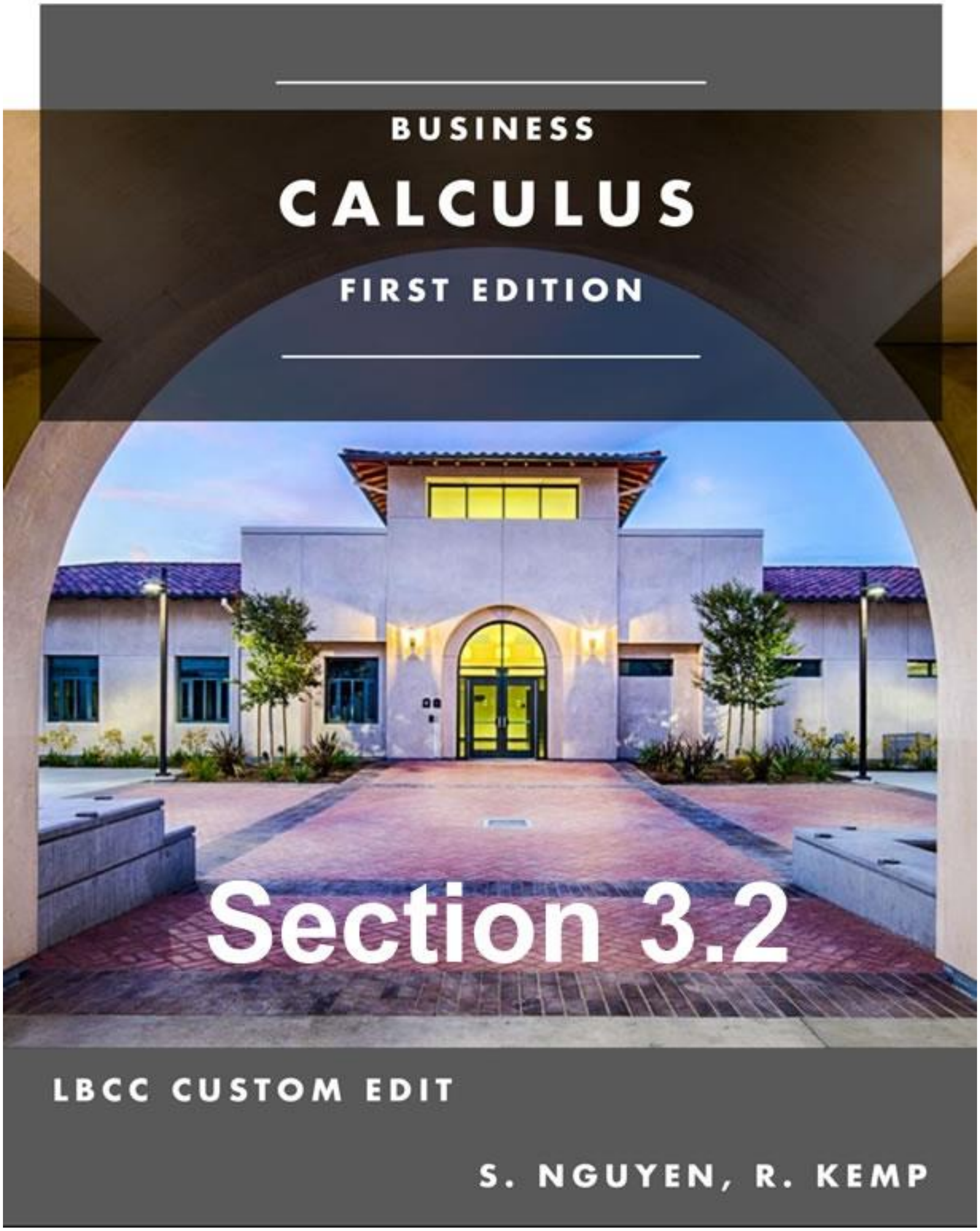
8. 6.99%

9. $417,710,242,922.55

10. 4.7%, 228.79 cents (=$2.29)

11. $w(t) = 1.60(1.0464)^t$, the model predicts $5.70, so the actual number is lower than the model predicts.

12. a) E, b) A, c) F, d) C, e) B, f) D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3.2 - LOGARITHMIC FUNCTIONS

Logarithms are the inverse of exponential functions; they allow us to undo exponential functions and solve for the exponent.

They are also commonly used to express quantities that vary widely in size.

Exponential & Logarithmic Function

Logarithm is the inverse function to an Exponential

The logarithm function ($\log_b(x)$), is equivalent to the value of the exponent, we raise the base (b), to get the value of (x).

$$y = f(x) = \log_b(x) \quad \rightarrow \quad x = b^y$$

The Natural logarithm function ($\ln(x)$), is equivalent to the value of the exponent, we raise the Euler constant ($e = 2.718282$), to get the value of (x).

$$y = f(x) = \ln_e(x) = \ln(x) \quad \rightarrow \quad x = e^y$$

Common and Natural Logarithms

The **Common** “log” is the logarithm with base ($b = 10$), and is typically written [$\log_{10}(x) = \log(x)$].

The **Natural** “log” is the logarithm with base ($b = e$), and is typically written [$\ln_e(x) = \ln(x)$].

However, as you probably know, or have noticed, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we’ll see briefly.

Since most exponential functions we are dealing with in this course will be natural exponential functions, with base e, the logarithmic functions that we will mostly deal with are the natural logarithmic functions ln(x).

We need one more formula to help us change from a logarithm of any base to a natural or common logarithm.

Properties of Logarithmic & Natural Logarithmic Functions

Logarithm Proprieties		Natural Logarithm Proprieties
$\log_b(ab) = \log_b(a) + \log_b(b)$	|	$\ln(ab) = \ln(a) + \ln(b)$
$\log_b\left(\frac{a}{b}\right) = \log_b(a) - \log_b(b)$	|	$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
$\log_b(a)^c = c \cdot \log_b(a)$	|	$\ln(a)^c = c \cdot \ln(a)$

Inverse Properties

Logarithm Proprieties		Natural Logarithm Proprieties
$x = \log_b(b^x)$	|	$x = \ln(e^x)$
$x = b^{\log_b(x)}$	|	$x = e^{ln(x)}$

Change of base formula

$$\log_a b = \frac{\log_c(b)}{\log_c(a)} = \frac{\log(b)}{\log(a)} = \frac{\ln(b)}{\ln(a)}$$

We can use this formula to change a logarithm of any base to any other base we wish, but we will use to change to natural log or a common log, whichever we prefer.

Commonly Used - Proprieties of Natural Logarithms:

1.	$\ln(0) = undefined$		
2.	$\ln(1) = 0$	6.	$\ln(ab) = \ln(a) + \ln(b)$
3.	$\ln(e) = 1$	7.	$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
4.	$e^{\ln(x)} = x$	8.	$\ln\left(\frac{1}{b}\right) = \ln(1) - \ln(b) = -\ln(b)$
5.	$\ln(e^x) = x$	9.	$\ln(a)^c = c \cdot \ln(a)$

Example Problem #1

Evaluate the following logarithmic functions:

a) Rewrite as a logarithm: $2^3 = 8$

Solution: $y = \log_2(8) = 3$

b) Rewrite in exponential form: $\log_5(125) = 3$

Solution: $x = 5^3 = 125$

c) Evaluate the function: $\log_{\frac{1}{2}}\left(\frac{1}{4}\right)$

Solution: $\left(\frac{1}{2}\right)^y = \frac{1}{4}$ $\quad Therefore \quad y = 2$

d) Evaluate the function: $\log_3(27)$

Solution: $(3)^y = 27$ $\quad Therefore \quad y = 3$

How can we evaluate logarithms that are not as clear as ($\log_2(8)$), let's say ($\log_2(10)$)? Since 10 is not a power of 2, like 8 was (= 2^3), we will need to ask a calculator for help.

Example Problem #2

Evaluate the following logarithmic functions:

a) Evaluate the function: $\log_2(10)$

Solution: $\log_2(10) = \frac{\ln(10)}{\ln(2)} \approx 3.3219$

b) Evaluate the function: $\log_4(307)^2$

Solution: $\log_4(307)^2 = 2\left(\frac{\ln(307)}{\ln(4)}\right) \approx 2(4.131) = 8.262$

Let's now solve some application that will require us to solve exponential or logarithmic equations. In either case, we will use the natural log to help us solve the problem.

Example Problem #3

In the last section, we predicted the population of India to grow to be 1.14 billion people in (t) years after 2008, by using the function ($f(t)$).

$$f(t) = 1.14 \cdot (1 + 0.0134)^t$$

If the population of India continues following this growing trend, when will the population reach 2 billion people?

Solution:

We want the population to equal 2 billion, so let the function equal [$f(t) = 2$].

$$f(t) = 1.14 \cdot (1 + 0.0134)^t = 2$$

$$(1 + 0.0134)^t = \frac{2}{1.14} = 1.7544$$

Next, we take the “Natural log” of both sides of the above equation.

$$\ln(1.0134)^t = \ln(1.7544)$$

$$t \cdot \ln(1.0134) = \ln(1.7544)$$

$$t = \frac{\ln(1.7544)}{\ln(1.0134)} \approx \frac{0.562127}{0.013311} \approx 42.23$$

$$t = 42.23 \ years$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Notice that we take the "Natural Log" both sides of the equation using a (ln).

This is the same method we are using when we square root, or we cube, and so on, both sides of an equation in order to isolate the radical.

Example Problem #4

Solve for the time (t) in years, in the following function:

$$5e^{-0.3t} = 2$$

Solution:

$$e^{-0.3t} = \frac{2}{5}$$

Next, we take the "Natural log" of both sides of the above equation.

$$\ln(e^{-0.3t}) = -0.3t \cdot \ln(e) = \ln\left(\frac{2}{5}\right)$$

$$-0.3t = \ln\left(\frac{2}{5}\right) = -0.9163$$

$$t = \frac{-0.9163}{-0.3} = 3.054$$

$$t = 3.054 \ years$$

While we don't often need to sketch the graph of a logarithm, it is helpful to understand the basic shape.

Graphical Features of the Logarithm

Graphically, in the function [$y = f(x) = \log_b(x)$]

The graph has a horizontal intercept at (1, 0), i.e. [$\log_b(1) = 0$]

The graph has a vertical asymptote at $x = 0$

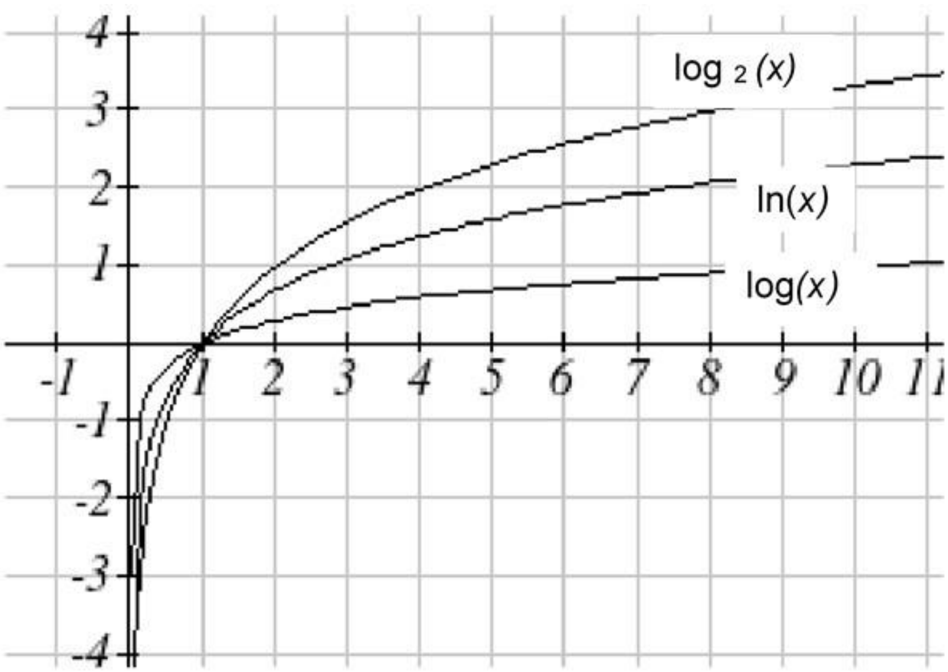
The graph is increasing and concave down

The domain of the function is $x > 0$, or $(0,\infty)$

The range of the function is all real numbers, or $(-\infty,\infty)$

When sketching a general logarithm with base (*b*), it can be helpful to remember that the graph will pass through the points (1, 0) and (*b*, 1).

To get a feeling for how the base affects the shape of the graph, examine the graphs below.



Another important observation made was the domain of the logarithm: $x > 0$.

Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Example Problem #5

Find the domain of the function:

$$f(x) = \ln(6 - 2x)$$

Solution:

The input into a logarithm, regardless of the base is only positive numbers, so

$$6 - 2x > 0$$

$$-2x > -6$$

$$x < \frac{-6}{-2} = 3$$

$$x < 3$$

So, the domain of the function is $(-\infty, 3)$ or $\{x \mid x < 3\}$.

We have listed the properties of logarithms in the beginning of the section, but haven't used them much. Here are the properties again, this time written for the natural logarithmic function.

Example Problem #6

Use the properties of logarithms to simplify the following functions:

a) The following function: $f(x) = \ln(5x)^2 - \ln\left(\frac{e}{x}\right) + \ln x^4$

Solution:

$$f(x) = 2 \cdot (\ln(5) + \ln(x)) - [\ln(e) - \ln x] + 4 \cdot \ln x$$

$$f(x) = 2 \cdot \ln(5) + 2 \cdot \ln x - \ln(e) + \ln x + 4 \cdot \ln x$$

$$f(x) = 2 \cdot \ln(5) + 7 \cdot \ln x - 1$$

b) The following function: $f(x) = e^{\ln x} - \ln(1) - \ln e^{7x} + 16x$

Solution:

$$f(x) = x - 0 - 7x + 16x = 10x$$

c) The following function: $f(x) = e^{\ln 5x} - \ln \sqrt[4]{e^5} + \ln e^{\frac{5}{4}} + 11x$

Solution:

$$f(x) = 5x - \ln\left(e^{\frac{5}{4}}\right) + \frac{5}{4}(\ln e) + 11x$$

$$f(x) = 5x - \frac{5}{4} + \frac{5}{4} + 11x = 16x$$

Example Problem #7

Calculating the Doubling Time for a Principle Investment – Using Periodic Compounding

A principle amount of investment is deposited in a bank at 14.3% interest compounded quarterly. How long will it take, for the principle to double in value?

Solution:

Using the "Future Value" compound interest formula:

$$P(t) = P\left(1 + \frac{r}{n}\right)^{n \cdot t} = 2P$$

- Principle value – $P = P$
- Rate in Percent – $r = 0.143$
- Periods per year – $n = 4$
- Time in # of Years – $t = ?$

$$P(t) = P\left(1 + \frac{0.143}{4}\right)^{4t} = 2P$$

$$(1.03575)^{4t} = 2$$

$$\ln(1.03575)^{4t} = \ln 2$$

$$4t \cdot \ln(1.03575) = \ln 2$$

$$t = \frac{\ln 2}{4 \cdot \ln(1.03575)} = 4.93 \ \ years$$

The amount of time it would take a principle investment at 14.3% interest compounded quarterly is, 4.9 years or approximately 5 years.

Triple in Value: $P\left(1 + \frac{r}{n}\right)^{n \cdot t} = 3P$

Increase by 50% in Value: $P\left(1 + \frac{r}{n}\right)^{n \cdot t} = 1.5P$

Example Problem #8

Calculating the Doubling Time for a Principle Investment – Using Continuous Compounding

A principle amount of investment is deposited in a bank at 12.9% interest compounded continuously. How long will it take, for the principle to double in value?

Solution:

Using the "Future Value" compound interest formula:

$$P(t) \;=\; Pe^{r \cdot t} \;=\; 2P$$

- Principle value – $P = P$
- Rate in Percent – $r = 0.129$
- Time in # of Years – $t \;= ?$

$$P(t) \;=\; Pe^{(0.129) \cdot t} \;=\; 2P$$

$$e^{(0.129) \cdot t} \;=\; 2$$

$$\ln e^{(0.129) \cdot t} \;=\; \ln 2$$

$$(0.129) \cdot t \cdot \ln(e) \;=\; \ln 2$$

$$t \;=\; \frac{\ln 2}{0.129} \;=\; 5.37 \;\; years$$

The amount of time it would take a principle investment at 12.9% interest compounded continuously is, 5.37 years or approximately 5.5 years.

Triple in Value: $P(t) \;=\; Pe^{r \cdot t} \;=\; 3P$

Increase by 50% in Value: $P(t) \;=\; Pe^{r \cdot t} \;=\; 1.5P$

Spreading of Information

The news-papers, radio stations, television stations, and people, spread news and information, to people in their community; who in turn, spread that information, to family, friends, and to those in their near location, cities, and towns.

Once the news or information starts to spread, the proportion of people that hear the news and information within (t) hours is given by the following equation.

Spreading of News & Information Formula:

$$P(t) = 1 - e^{-kt} = \%\binom{Percent\ of\ people}{hear\ news}$$

Example Problem #9

A hurricane warning is broadcast to the public, nearby cities, and towns. The proportion of people who hear the news within (t) hours, of its initial broadcast is $(P(t))$:

$$P(t) = 1 - e^{-0.45t}$$

When will 65% of the people have heard the news?

Solution:

Using the Spreading of News & Information Formula:

$$P(t) = 1 - e^{-0.45t} = 0.65$$

$$e^{-0.45t} = (1 - 0.65) = 0.35$$

$$\ln e^{-0.45t} = \ln(0.35)$$

$$(-0.45t) \cdot \ln e = \ln(0.35)$$

$$t = \frac{\ln(0.35)}{-0.45} = 2.33\ hours$$

Therefore, it takes about 2.33 hours for about 65% of the people in nearby cities and towns to hear the news.

3.2 - EXERCISES

Solve the Logarithmic and Exponential functions			
1.	Rewrite in exponential form. $ln(w) = n$	2.	Rewrite in logarithmic form. $e^k = h$
3.	Solve for x $e^{5x} = 17$	4.	Solve for t $10e - 0.03^t = 4$
5.	Solve for x $10 - 8\left(\frac{1}{2}\right)^x = 5$	6.	Solve for t $2\,(1.08)^{4t} = 7$
7.	Find the domain of the functions: a) $f(x) = ln\,(12 - 3x)$ b) $g(x) = ln\,(2x + 12)$	8.	Use the properties of exponents to simplify the function $f(x) = ln(4x^2) - lne - ln\frac{x}{2}$
9.	Use the properties of logs to simplify the function $f(x) = 2x + ln(1) - ln(e^{4x})$	10.	Use the properties of logs to simplify the function $f(x) = ln(x^3) - 3ln(x) + ln(e)$

11.	According to the World Bank, at the end of 2013 (t = 0) the U.S. population was 316 million and was increasing according to the following model: $$P(t) = 316e^{.0074t},$$ where P is measured in millions of people and t is measured in years after 2013. Determine when the U.S. population will be twice what it was in 2013.
12.	The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $$A(t) = 1000(1.04)^t.$$ Determine how long it takes for the original investment to triple.
13.	If \$1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to \$1500? How long will it take if the money is compounded continuously?
14.	The amount in an account grows at a rate of 8% compounded continuously. How long will it take for the amount to double? How long will it take for the amount to increase by 50%?
15.	The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

16.	A company begins and ad advertising for its product. The percentage of people in the city who got to see the ad t days from the start of the ad campaign is: $$f(t) = 1 - e^{-0.06t}$$ How long will it take until the ad reaches 50% of the people in town.
17.	The advertising response N(a), where N is the number of items sold and a is the amount spent on advertising in thousands of dollars, is $$N(a) = 1000 + 400ln(a)$$ for $a > 0$ a) How many units were sold after spending $4000 in advertising? b) How much money should be spent in order to sell 1500 units?

Solutions:

1. $e^n = w$

2. $ln\,(h) = k$

3. 0.5667

4. -0.8964

5. 0.678

6. 4.069

7. $a)$ $(-\infty, 4)$, b) $(-6, \infty)$

8. $ln(4) + ln(2) + ln(x) - 1 = ln(8) + ln(x) - 1$

9. $-2x$

10. 1

11. By the beginning of the year 2107 (93.67 years later than end of 2013)

12. 28 years

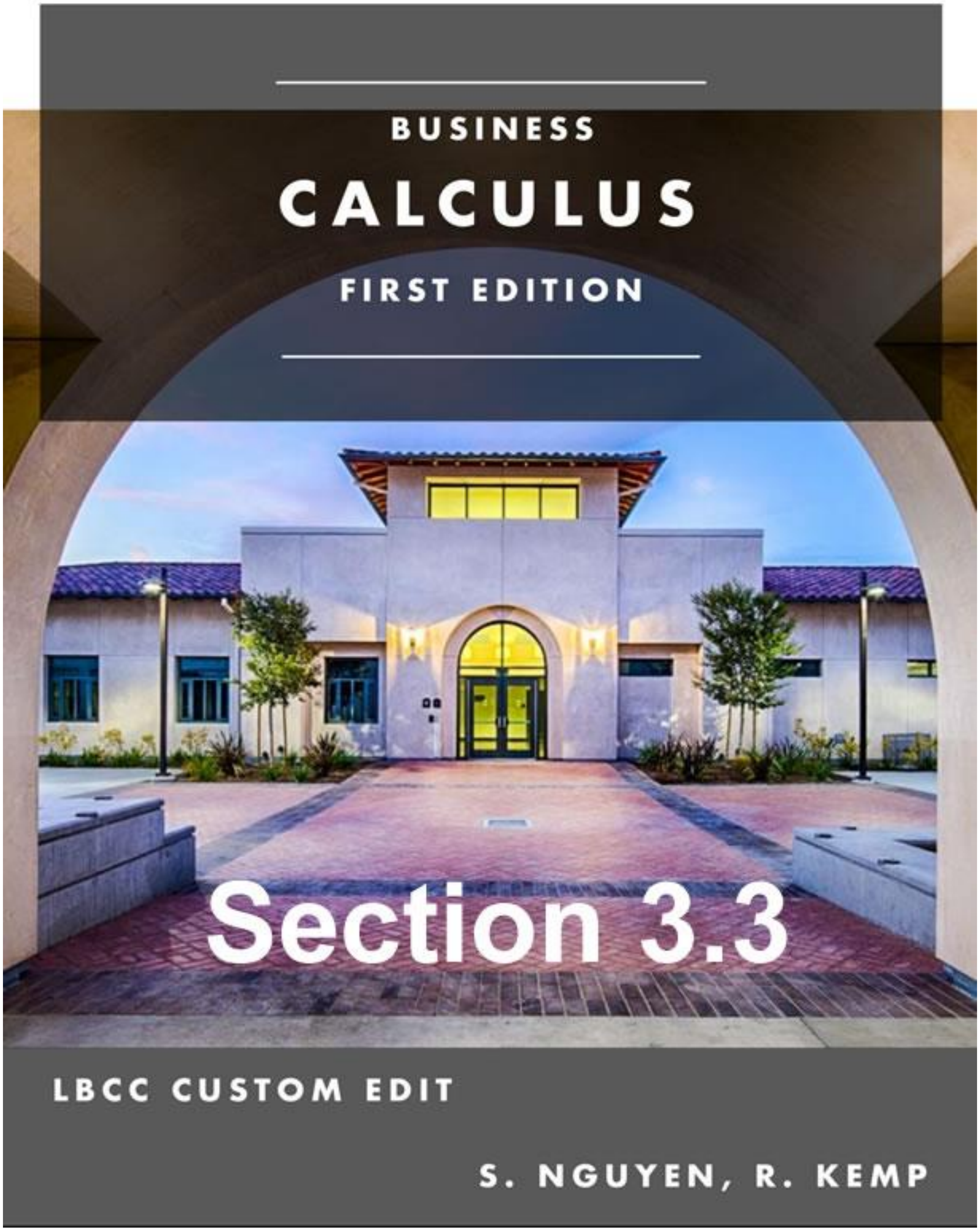
13. 13.53 years and 13.51 years

14. 8.66 years and 5.07 years

15. By year 2013

16. 11.55 days

17. a) about 4,318 units, b) 3.490 thousand dollars, or $3,490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3.3 - DERIVATIVES OF THE NATURAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Now it's time to introduce the derivatives of the two functions we just reviewed. Since we will focus on only applications of the natural exponential and logarithmic functions, we will only study the rules for taking the derivatives of these functions.

The Derivative of the Natural Exponential Function

The **Natural Exponential Function**:

$$f(x) = e^x$$

The **derivative of the Natural Exponential Function**:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}e^x = e^x$$

We can extend this formula to the derivative of a composition of a natural exponential function with another function by using the chain rule:

Given the composition Function:

$$f(x) = e^{g(x)}$$

The **derivative of this composition Function is**:

$$f'(x) = (e^{g(x)})' = \frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot \frac{d}{dx}g(x)$$

$$f'(x) = (e^{g(x)})' = g'(x) \cdot e^{g(x)}$$

Example Problem #1

Find the derivative of the Exponential function:

$$f(x) = e^{x^3}$$

Solution:

Now, let the function be defined:

$$f(x) = e^{g(x)}$$

$$g(x) = x^3 \quad ; \quad g'(x) = 3x^2$$

The derivative of the exponential function stays the same, then it must be multiplied by the derivative of the exponent.

$$f'(x) = \frac{d}{dx}\left(e^{g(x)}\right) = g'(x) \cdot e^{g(x)}$$

$$f'(x) = \frac{d}{dx}\left(e^{x^3}\right) = 3x^2 \cdot e^{x^3}$$

Example Problem #2

Find the derivative of the Exponential function:

$$f(x) = e^{\left(\frac{x^4}{4} - 2x^2 + 1\right)}$$

Solution:

Now, let the function be defined:

$$f(x) = e^{g(x)}$$

$$g(x) = \frac{x^4}{4} - 2x^2 + 1 \quad ; \quad g'(x) = x^3 - 4x$$

The derivative of the exponential function stays the same, then it must be multiplied by the derivative of the exponent.

$$f'(x) = \frac{d}{dx}\left(e^{g(x)}\right) = g'(x) \cdot e^{g(x)}$$

$$f'(x) = \frac{d}{dx}\left(e^{\left(\frac{x^4}{4} - 2x^2 + 1\right)}\right) = (x^3 - 4x)e^{\left(\frac{x^4}{4} - 2x^2 + 1\right)}$$

Example Problem #3

Find the derivative of the Exponential function:

$$f(x) = x^2 \cdot e^{3x}$$

Solution:

In this case, we must use the product rule as well as the chain rule:

$$f'(x) = \frac{d}{dx}(x^2 \cdot e^{3x}) = e^{3x} \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}(e^{3x})$$

$$f'(x) = e^{3x} \cdot [2x] + x^2 \cdot \left[e^{3x} \cdot \frac{d}{dx}(3x)\right]$$

$$f'(x) = 2x \cdot e^{3x} + 3x^2 \cdot e^{3x}$$

$$f'(x) = e^{3x}(2x + 3x^2) = xe^{3x}(2 + 3x)$$

Example Problem #4

Find the derivative of the Exponential function:

$$f(x) = (x - 1)^2 \cdot e^{(7x-6)}$$

Solution:

In this case, we must use the product rule as well as the chain rule:

$$f'(x) = \frac{d}{dx}\left[(x - 1)^2 \cdot e^{(7x-6)}\right]$$

$$f'(x) = e^{(7x-6)} \cdot \frac{d}{dx}[(x - 1)^2] + (x-1)^2 \cdot \frac{d}{dx}\left(e^{(7x-6)}\right)$$

$$f'(x) = e^{(7x-6)} \cdot [2(x - 1)] + (x-1)^2 \cdot \left(7e^{(7x-6)}\right)$$

$$f'(x) = 2(x - 1)e^{(7x-6)} + 7(x-1)^2 \cdot e^{(7x-6)}$$

$$f'(x) = (x - 1)e^{(7x-6)} \cdot [2 + 7(x-1)]$$

$$f'(x) = (x - 1)e^{(7x-6)} \cdot [2 + 7x - 7]$$

$$f'(x) = (x - 1)e^{(7x-6)} \cdot [7x - 5]$$

$$f'(x) = (x - 1)(7x - 5) \cdot e^{(7x-6)}$$

Example Problem #5:

Find the derivative of the Exponential function:

$$f(x) = \frac{e^{(2x^2+1)}}{2 + 4x^2}$$

Solution:

In this case, we must use the Quotient rule or the chain rule:

$$f'(x) = \frac{d}{dx}\left(\frac{e^{(2x^2+1)}}{2 + 4x^2}\right)$$

$$= \frac{(2+4x^2)\cdot\frac{d}{dx}\left(e^{(2x^2+1)}\right) - e^{(2x^2+1)}\cdot\frac{d}{dx}(2+4x^2)}{(2+4x^2)^2}$$

$$f'(x) = \frac{(2+4x^2)\cdot\left[e^{(2x^2+1)}\frac{d}{dx}(2x^2+1)\right] - e^{(2x^2+1)}\cdot(8x)}{(2+4x^2)^2}$$

$$f'(x) = \frac{(2+4x^2)\cdot\left[e^{(2x^2+1)}\cdot(4x)\right] - e^{(2x^2+1)}\cdot(8x)}{(2+4x^2)^2}$$

$$f'(x) = \frac{4x\cdot(2+4x^2)\cdot e^{(2x^2+1)} - 8x\cdot e^{(2x^2+1)}}{(2+4x^2)^2}$$

$$f'(x) = \frac{4x\cdot e^{(2x^2+1)}\cdot(2 + 4x^2 - 2)}{(2+4x^2)^2}$$

$$f'(x) = \frac{16x^3\cdot e^{(2x^2+1)}}{(2+4x^2)^2}$$

Now it's time to introduce the derivative of the natural logarithmic functions.

The Derivative of the Natural Logarithmic Function

The **Natural Logarithmic Function**:

$$f(x) = \ln x$$

The **derivative of the Natural Logarithmic Function**:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(\ln x) = \frac{1}{x} = x^{-1}$$

We can extend this formula to the derivative of a composition of a natural logarithmic function with another function by using the chain rule:

Given the **composition Function**:

$$f(x) = \ln(g(x))$$

The **derivative of this composition Function**:

$$f'(x) = \frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)} \cdot \frac{d}{dx}g(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

Given the **composition Function**:

$$f(x) = \ln(g(x))^K = K \cdot \ln(g(x))$$

The **derivative of the Logarithmic Function**:

$$f'(x) = \left[\frac{d}{dx}K \cdot \ln(g(x))\right] = K \cdot \left[\frac{\frac{d}{dx}g(x)}{g(x)}\right] = K \cdot \left[\frac{g'(x)}{g(x)}\right]$$

Example Problem #6

The Total Annual Revenue ($R(t)$) that a business incurs selling **Designer Sport Coats** is given by the "Logarithmic" function in ten thousands of dollars.

The variable for time (t) is years.

$$R(t) = \ln(t^2 + 5) \quad \rightarrow \quad ten\ thousands$$

a) Find the instantaneous rate of change in the revenue or Marginal Revenue?

b) What is the Total Annual Reveue ($R(t)$) in the first year ($t = 1$), and in the third year ($t = 3$)?

c) Find the instantaneous rate of change in the revenue or Marginal Revenue in the first year ($t = 1$), and in the third year ($t = 3$)?

Solution:

a) Taking the derivative of this function, we need to use the chain rule. Be aware of which one is the outside function and which one the inside function.

$$R(t) = \ln\big(g(t)\big)$$

$$g(t) = t^2 + 5 \quad ; \quad g'(t) = 2t$$

$$R'(t) = MR(t) = \frac{d}{dt}\ln\big(g(t)\big) = \frac{g'(t)}{g(t)}$$

$$R'(t) = MR(t) = \frac{d}{dt}\ln(t^2 + 5) = \frac{1}{t^2 + 5} \cdot \frac{d}{dt}(t^2 + 5)$$

$$R'(t) = MR(t) = \frac{1}{t^2 + 5} \cdot 2t$$

The instantaneous rate of change in the revenue or Marginal Revenue function:

$$R'(t) = MR(t) = \frac{2t}{t^2 + 5} \quad \rightarrow \quad ten\ thousands/year$$

Example Problem #6 – Cont'd

b) the Revenue is given as, $R(t) = \ln(t^2 + 5) \rightarrow ten\ thousands$

After the first year ($t = 1$) the Yearly Revenue is given below:

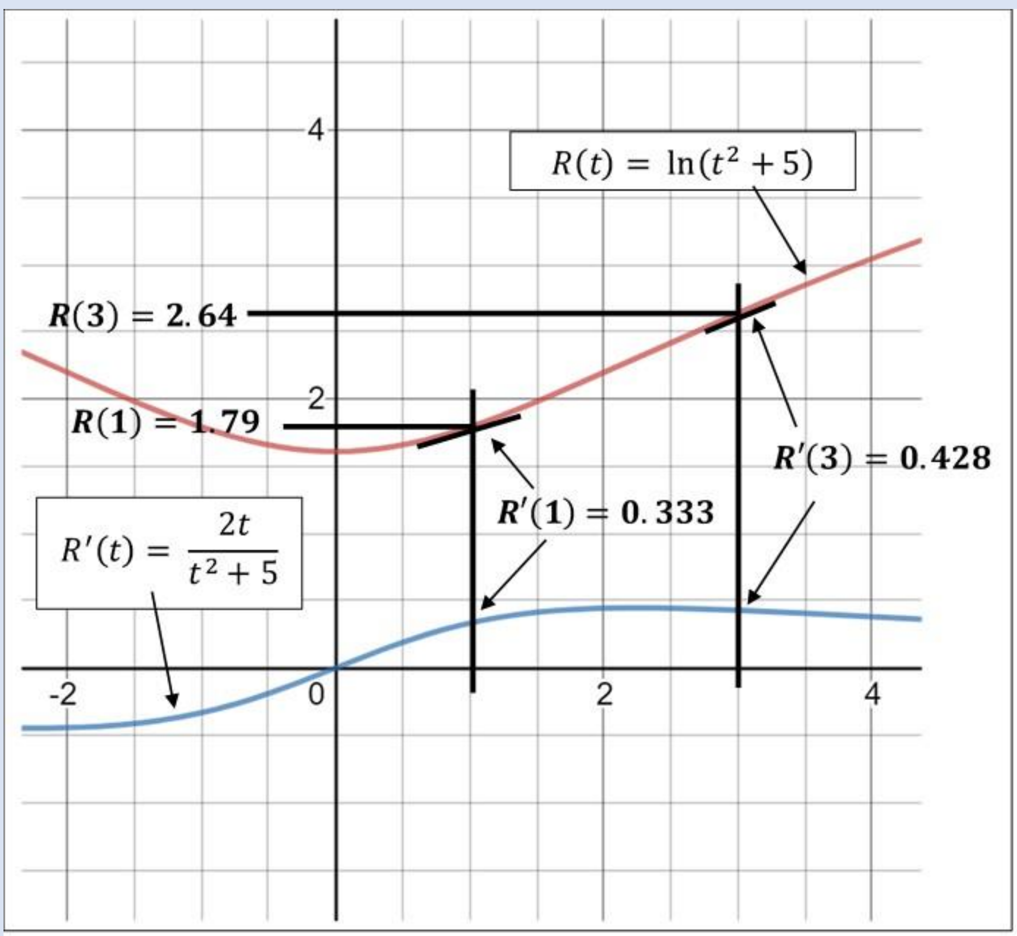
$$R(1) = \ln((1)^2 + 5) = \ln(6) = 1.79\ ten\ thousands$$

$$R(1) = 1.79(\$10{,}000) = \$17{,}900$$

After the third year ($t = 3$) the Yearly Revenue is given below:

$$R(3) = \ln((3)^2 + 5) = \ln(14) = 2.6391\ ten\ thousands$$

$$R(3) = 2.6391(\$10{,}000) = \$26{,}391$$



Example Problem #6 – Cont'd

c) the Marginal Revenue is given as, $MR(t) = \frac{2t}{t^2+5} \rightarrow \$10{,}000/year$

After the first year ($t = 1$) the Marginal Revenue is given below:

$$R'(1) = MR(1) = \frac{2(1)}{(1)^2 + 5} = \frac{1}{3} = 0.33333\ (\$10{,}000/year)$$

$$R'(1) = MR(1) = 0.3(\$10{,}000) = \$3{,}333.33\ \ dollars/year$$

After the first year the rate of change in the annual revenue or Marginal Revenue are $3,333,33 dollars/year.

After the third year ($t = 3$) the Marginal Revenue is given below:

$$R'(3) = MR(3) = \frac{2(3)}{(3)^2 + 5} = \frac{3}{7} = 0.42857\ (\$10{,}000/year)$$

$$R'(3) = MR(3) = 0.42857(\$10{,}000) = \$4{,}285.70\ dollars/year$$

After the third year the rate of change in the annual revenue or Marginal Revenue are $4,285 dollars/year.

Therefore, the Annual Total Revenue is increasing yearly at a natural logarithmic rate. The Annual Total Revenue increased from year one to year three.

$$\Delta R = R(3) - R(1) = (\$26{,}391 - \$17{,}900) = \$8491.00$$

And the rate of change in the Annual Total Revenue or Marginal Revenue is also increasing yearly. Therefore, the annual revenue is accelerating.

$$\Delta MR = MR(3) - MR(1) = (\$4{,}285.70 - \$3{,}333.33)\frac{dollars}{year}$$

$$\Delta MR = R(3) - R(1) = \$952.34\ dollars/year$$

Example Problem #7

Find the derivative ($f'(x)$) of the Logarithmic function:

$$f(x) = \sqrt{\ln(x+1)}$$

Solution:

First, let's express the above function in the form with an exponent, and let's apply the rules:

$$f(x) = (\ln(x+1))^{\frac{1}{2}}$$

Taking the derivative of this function, we need to use the chain rule. Be aware of which one is the outside function and which one the inside function

$$f(x) = (g(x))^{\frac{1}{2}}$$

$$g(x) = \ln(x+1) \quad ; \quad g'(x) = \frac{1}{x+1}$$

Next, take the derivative of the composition of functions.

$$f'(x) = \frac{1}{2}\left(g(x)\right)^{-\frac{1}{2}} \cdot g'(x)$$

$$f'(x) = \frac{1}{2}(\ln(x+1))^{-\frac{1}{2}} \cdot \frac{1}{(x+1)}$$

$$f'(x) = \frac{1}{2(x+1)\sqrt{\ln(x+1)}}$$

Example Problem #8

Find the derivative of the logarithmic function:

$$f(x) = ln\left(\frac{x^2}{2x + 1}\right)$$

Solution:

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$f(x) = \ln x^2 - \ln(2x + 1)$$

$$f(x) = 2 \cdot \ln x - \ln(2x + 1)$$

Now we do not have to deal with the quotient rule anymore.

Next, take the derivative of the above function:

$$f'(x) = \frac{d}{dx}[2 \cdot \ln x - \ln(2x + 1)]$$

$$f'(x) = 2 \cdot \frac{d}{dx}[\ln x] - \frac{d}{dx}\ln(2x + 1)$$

$$f'(x) = 2 \cdot \left(\frac{1}{x}\right) - \frac{\frac{d}{dx}(2x + 1)}{2x + 1}$$

$$f'(x) = \frac{2}{x} - \frac{2}{2x + 1}$$

Or

$$f'(x) = \frac{2 \cdot (2x + 1) - 2x}{x \cdot (2x + 1)} = \frac{2x + 2}{x \cdot (2x + 1)}$$

$$f'(x) = \frac{2 \cdot (x + 1)}{x \cdot (2x + 1)}$$

Example Problem #8 – Cont'd

Find the derivative of the logarithmic function using the Quotient Rule:

$$f(x) = ln\left(\frac{x^2}{2x + 1}\right)$$

Solution:

Next, take the derivative of the above function:

$$f'(x) = \frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

$$f'(x) = \frac{d}{dx}\left[ln\left(\frac{x^2}{2x + 1}\right)\right]$$

$$f'(x) = \frac{\frac{d}{dx}\left(\frac{x^2}{2x + 1}\right)}{\left(\frac{x^2}{2x + 1}\right)}$$

$$f'(x) = \frac{1}{\left(\frac{x^2}{2x + 1}\right)}\left[\frac{(2x + 1)\cdot\frac{d}{dx}(x^2) - x^2\cdot\frac{d}{dx}\cdot(2x + 1)}{(2x + 1)^2}\right]$$

$$f'(x) = \left(\frac{2x + 1}{x^2}\right)\cdot\left[\frac{(2x + 1)\cdot(2x) - x^2\cdot(2)}{(2x + 1)^2}\right]$$

$$f'(x) = \left(\frac{2x + 1}{x^2}\right)\cdot\left[\frac{4x^2 + 2x - 2x^2}{(2x + 1)^2}\right] = \left(\frac{2x + 1}{x^2}\right)\cdot\left[\frac{2x^2 + 2x}{(2x + 1)^2}\right]$$

$$f'(x) = \frac{2x + 2}{x\cdot(2x + 1)}$$

$$f'(x) = \frac{2\cdot(x + 1)}{x\cdot(2x + 1)}$$

Example Problem #9

a) Find the derivative ($f'(x)$) of the Logarithmic function:

$$f(x) = \ln\left(\sqrt{x+1}\right)$$

b) Find the tangent line of the Logarithmic function when ($x = 3$):

Solution:

First, let's express the above function in the form with an exponent, and let's apply one of the logarithmic rules:

$$f(x) = \ln(x+1)^{\frac{1}{2}} = \frac{1}{2} \cdot \ln(x+1)$$

Now, let the function be defined:

$$f(x) = \ln\left(g(x)\right)$$

$$g(x) = (x+1)^{\frac{1}{2}} \quad ; \quad g'(x) = \frac{(x+1)^{-\frac{1}{2}}}{2} = \frac{1}{2 \cdot (x+1)^{\frac{1}{2}}}$$

Next, take the derivative of the logarithmic function.

$$f'(x) = \frac{d}{dx}\ln\left(g(x)\right) = \frac{g'(x)}{g(x)}$$

$$f'(x) = \frac{d}{dx}\ln(x+1)^{\frac{1}{2}}$$

$$f'(x) = \frac{\frac{d}{dx}(x+1)^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}} = \frac{\frac{1}{2}(x+1)^{-\frac{1}{2}}}{(x+1)^{\frac{1}{2}}}$$

$$f'(x) = \frac{(x+1)^{-1}}{2} = \frac{1}{2 \cdot (x+1)}$$

Example Problem #9 – Cont'd

Now, let's use the other form of the function.

$$f(x) = \ln(x+1)^{\frac{1}{2}} = \frac{1}{2} \cdot \ln(x+1)$$

Now, let the function be defined:

$$f(x) = \frac{1}{2} \cdot \left[\ln\left(g(x)\right)\right]$$

$$g(x) = x + 1 \quad ; \quad g'(x) = 1$$

a) Taking the derivative of the logarithmic function:

$$f'(x) = \frac{1}{2} \cdot \left[\frac{d}{dx} \ln\left(g(x)\right)\right] = \frac{1}{2} \cdot \left[\frac{g'(x)}{g(x)}\right]$$

$$f'(x) = \frac{1}{2} \cdot \left[\frac{d}{dx} \ln(x+1)\right]$$

$$f'(x) = \frac{1}{2} \cdot \left[\frac{\frac{d}{dx}(x+1)}{x+1}\right]$$

$$f'(x) = \frac{1}{2 \cdot (x+1)} = \frac{(x+1)^{-1}}{2}$$

a) Find the tangent line of the Logarithmic function when ($x = 3$):

First find the value of the slope ($m = f'(3)$)

$$m = f'(x) = \frac{1}{2 \cdot (x+1)}$$

$$m = f'(3) = \frac{1}{2 \cdot (3+1)} = \frac{1}{8} = 0.125$$

Example Problem #9 – Cont'd

Next find the y-point on curve of the logarithmic function ($f(x)$) where ($x = 3$)

$$x_1 = 3$$

$$y_1 = f(x_1) = \ln\left(\sqrt{x + 1}\right)$$

$$y_1 = f(3) = \ln\left(\sqrt{3+1}\right) = \ln(2) = 0.693147$$

Next use the point slope form of the equation to find the tangent line of the logarithmic function.

$$y - y_1 = m(x - x_1)$$

$$y - 0.693147 = 0.125(x - 3)$$

$$y = 0.125x - 0.375 + 0.693147$$

$$y = 0.125x + 0.318147$$

Or

$$y = \frac{1}{8}x + 0.318147$$

Or

$$y = \frac{1}{8}x + \left(\frac{3}{8} + \ln(2)\right)$$

Now, let's look at some applications of derivatives. These are no different from the applications we discussed in chapters 1 and 2, except for we are now dealing with exponential and logarithmic functions as well.

Example Problem #10

A principal amount of money of $10,000 grows continuously at an interest rate of 4%. Find the ***rate of growth*** after 4 years.

Solution:

Since the money grows continuously, the amount after t years will be given by the formula:

$$P(t) = P \cdot \mathrm{e}^{rt}$$

- Principle value – $P = \$10{,}000$
- Rate in Percent – $r = 0.04$
- Time in # of Years – $t = 4\ years$

$$P(t) = 10{,}000 \cdot e^{0.04t}$$

The rate of growth will be given by the derivative:

$$P'(t) = 10{,}000e^{0.04t} \cdot 0.04 = 400e^{0.04t}$$

After four (4) years ($t = 4\ years$):

$$P'(4) = 400 \cdot e^{0.04(4)} = \$469.40$$

This means that after 4 years, the rate of growth of the amount in the account will be ($P'(t) = \$469.40$) dollars per year.

Now let's look at some additional, derivative techniques using "Generalized" exponent and logarithmic functions.

The Derivative of the "Generalized" Exponential Function

The **Exponent Function** (where $[b > 0]$):

$$f(x) = b^x$$

The **derivative of the Logarithmic Function**:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}b^x = (\ln b) \cdot b^x$$

We can extend this formula to the derivative of a composition of an exponential function with another function by using the chain rule:

The **Exponent Function** (where $[b > 0]$):

$$f(x) = b^{g(x)}$$

The **derivative of the Exponent Function**:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}b^{g(x)} = (\ln b) \cdot b^{g(x)} \cdot \left[\frac{d}{dx}g(x)\right]$$

$$f'(x) = (\ln b) \cdot g'(x) \cdot f(x) = (\ln b) \cdot g'(x) \cdot b^{g(x)}$$

The derivative, “Generalized” logarithmic functions.

The Derivative of the “Generalized” Logarithmic Function

The **Logarithmic Function**:

$$f(x) = \log_b x$$

The **derivative of the Logarithmic Function**:

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \log_b x = \frac{1}{(\ln b) \cdot x}$$

We can extend this formula to the derivative of a composition of a logarithmic function with another function by using the chain rule:

The **Logarithmic Function**:

$$f(x) = \log_b g(x)$$

The **derivative of the Logarithmic Function**:

$$f'(x) = \frac{d}{dx} f(x)$$

$$f'(x) = \frac{d}{dx} \log_b g(x) = \frac{\frac{d}{dx} g(x)}{(\ln b) \cdot g(x)} = \frac{g'(x)}{(\ln b) \cdot g(x)}$$

3.3 - EXERCISES

Find the derivatives of the following functions.			
1.	$f(x) = e^{2x}$	2.	$f(x) = ln(5x)$
3.	$f(x) = x^3 - 5x + e^{x+3}$	4.	$f(x) = ln(x^2)$
5.	$f(x) = x^4 ln(x) - 4x^4$	6.	$f(x) = \ln\left(\frac{x+4}{x^3}\right)$
7.	$f(x) = x^3 e^x + ln(1)$	8.	$f(x) = ln(e^x)$
9.	$f(x) = ln(e^x - 3x)$	10.	$f(x) = ln(x\sqrt{x+2})$
11.	$f(x) = 1 - e^{-0.087x}$	12.	$f(x) = \sqrt{e^{2x} + 3x - 1}$
13	$f(x) = \ln\left(\frac{x^4 + 2x^3}{x^2}\right)^5$	14	$f(x) = \ln\left(\frac{e^{2x} + 1}{2x}\right)$
15.	$f(x) = x^2 e^{(2x-5)}$	16.	$f(x) = 2e^{\left(\frac{x^2}{2} - x\ln x\right)}$

17.	The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a continuous rate of 5%. a. Write the exponential function that relates the total population as a function of t. b. Use it to determine the instantaneuos rate at which the population is changing in year t. c. Determine the rate at which the population is changing in 10 years and interpret this answer.
18.	The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961is modeled by the function $$N(t) = 5.3e^{0.093t^2-0.87t},$$ ($0 \leq t \leq 4$), where N(t) gives the number of cases (in thousands) and t is measured in years, with t = 0 corresponding to the beginning of 1960. a. Evaluate N(0) and N(4). Briefly describe what these values indicate about the disease in New York City. b. Show work that evaluates N′ (0) and N′ (4) and interpret your answers.
19.	A $25,000 car depreciates every year according to the formula $$V(t) = 25000e^{-0.15t}$$ where V is the value of the car in dollars, and t is the time, in years, since the car has been purchased. a) Find the instantaneous rate of change in the value of the car when it is new. b) Find the instantaneous rate of change in the value of the car 10 years after it's been purchased.

20.	The concentration C in mg, of a medication in the body, t hours after it has been taken, is given by $$C(t) = 20\text{t}\, e^{-t}$$ a) Find the concentration 2 hours later. b) Find the rate of change in the concentration. c) When will the concentration reach its maximum?
21.	The temperature of an object in degrees Fahrenheit after t hours is given by the equation $$T(t) = 68e^{-0.174t} + 72$$ a) What is the temperature of the object initially and one hour later? b) What is the instantaneous rate of change in the temperature initially and one hour later?
22.	The amount of items N being sold after an amount a is spent on advertising in thousands of dollars, is $$N(a) = 1000 + 400ln(a),$$ a ≥ 1. a) Find the rate of change in the number of items sold and evaluate it at $1000 spent. b) Is there a minimum or a maximum value?
23.	Find the equation of the tangent line to $f(x) = e^{2x+4}$ when x = - 2.
24.	Find the equation of the tangent line to $f(x) = e^{2x+4}$ when x = 2.

Solutions:

1. $f'(x) = 2e^{2x}$

2. $f'(x) = \frac{1}{x}$

3. $f'(x) = 3x^2 - 5 + e^{x+3}$

4. $f'(x) = \frac{2}{x}$

5. $f'(x) = 4x^3 ln(x) - 15x^3$

6. $f'(x) = \frac{1}{x+4} - \frac{3}{x}$

7. $f'(x) = 3x^2e^x + x^3e^x$

8. $f'(x) = 1$

9. $f'(x) = \frac{e^x - 3}{e^x - 3x}$

10. $f'(x) = \frac{1}{x} + \frac{1}{2(x+2)}$

11. $f'(x) = 0.087e^{-0.087x}$

12. $f'(x) = \frac{2e^{2x}+3}{2\sqrt{e^{2x}+3x-1}}$

13. $f'(x) = \frac{10\,(2x+3)}{x\,(x+2)} - \frac{10}{x} = \frac{10}{x}\left(\frac{2x+3}{x+2} - 1\right) = \frac{10}{x}\left(\frac{x+1}{x+2}\right)$

14. $f'(x) = \frac{2e^{2x}}{e^{2x}+1} - \frac{1}{x} = \frac{e^{2x}(2x-1)-1}{x(e^{2x}+1)}$

15. $f'(x) = 2x(1 + x)e^{(2x-5)}$

16. $f'(x) = 2e^{\left(\frac{x^2}{2} - x\ln x\right)}(x - lnx - 1)$

17. a) $P(t) = 500{,}000e^{0.05t}$, $b)$ $25{,}000e^{0.05t}$, c) The population will be increasing by 41,218 people per year by 2010.

18. a) $N(0) = 5.3, N(4) = 0.723$. The number of cases of influenza in NYC in 1960 was 5,300, while in 1964 it was 723 cases.

b) $N'(t) = 5.3e^{0.093t^2-0.87t}(0.186t - 0.87), N'(0) = -4.611, N'(4) = -0.091$. After 1960, the number of cases of influenza was declining by 4,611 cases per year, while after 1964 the number of cases was declining by 91 cases per year.

19. $V'(t) = -3750e^{-0.15t}$, $V'(0) = -3750,\ V'(10) = -836.74$. The value of the car is decreasing by $3750/year in the first year and by $836.74 after 10 years.

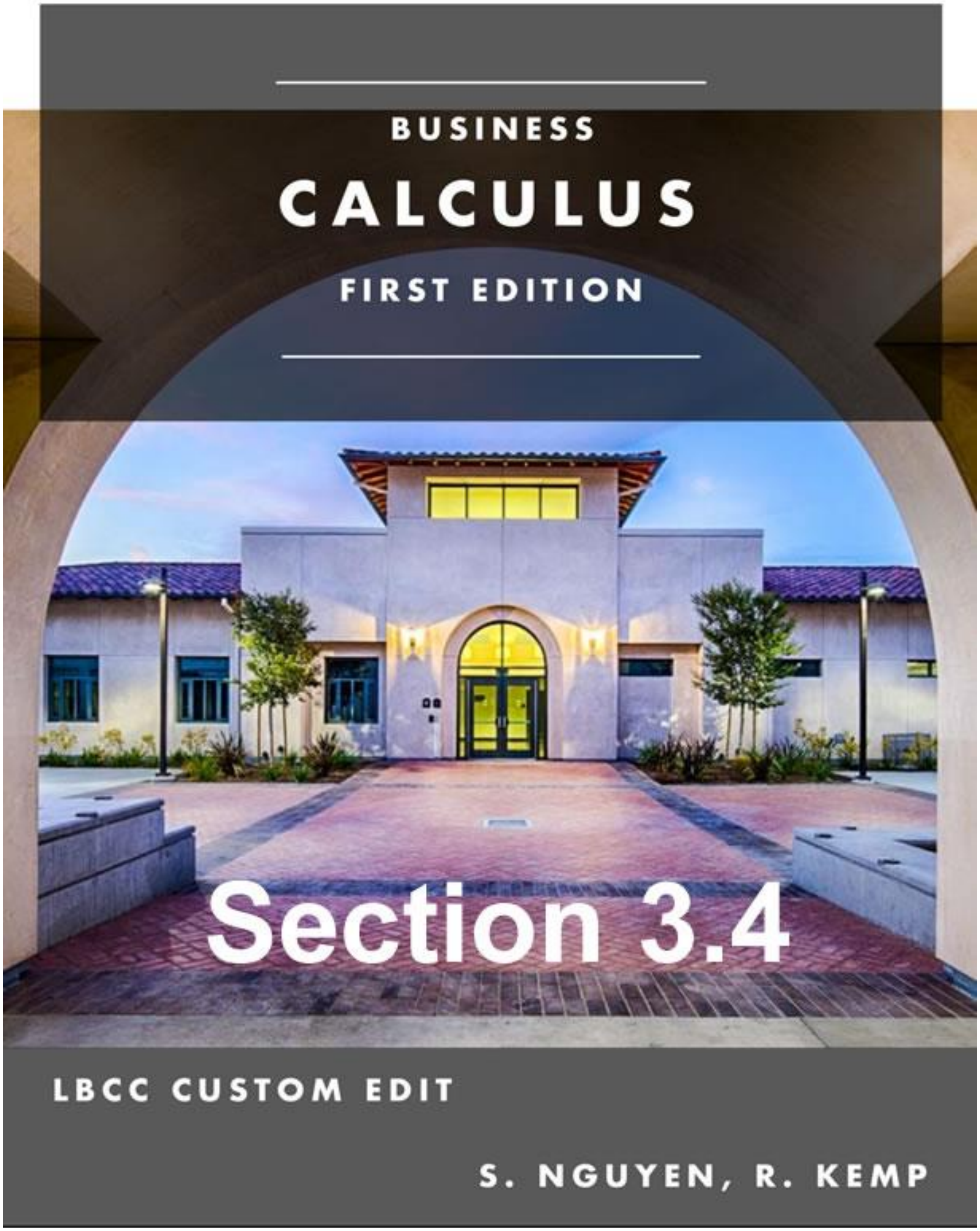
20. a) 5.41 mg, b) $C'(t) = 20e^{-t}(1 - t)$, c) after 1 hour.

21. a) 140° F and 129.14° F, b) -11.82°F/hour and -9.94°F/hour

22. a) $N'(1) = 400$ items/one thousand dollars,
b) The minimum is 1000 units, when 1 thousand dollars are spent, and there is no maximum.

23. $y = 2x + 5$

24. $y = e^8(2x - 3)$


BUSINESS
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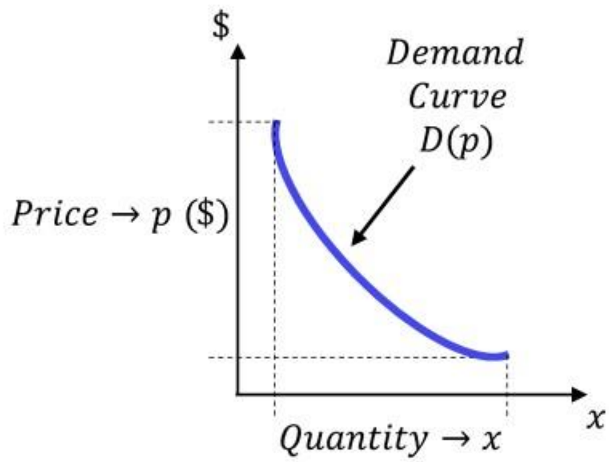
S. NGUYEN, R. KEMP

3.4 - ELASTICITY OF DEMAND

We know that demand functions are decreasing, i.e. when the price increases, the quantity demanded goes down.



All goods, products and services have a certain price "Demand $D(p)$" for that item, where the larger the quantity of items available the price gets smaller or goes lower.

And inversely the smaller the quantity of items available for that same good, product or service, the price "Demand $D(p)$" goes up or gets larger.

Together the price "Demand $D(p)$" in number of items, multiplied by the actual price (p) per item is a measure of revenue termed "Consumer Expenditure".

$$R(p) = \text{(number of items)} \cdot \text{(price/item)}$$

$$R(p) = p \cdot D(p)$$

The "Consumer Expenditure" is equal to a measure of "Revenue".

Consumer Expenduture will allow us to measure the Revenue $R(p)$ when the price increases or decreases due to changes in demand.

Example Problem #1

Let the Demand function ($D(p)$) for the amount of a certain pair of shoes, a consumer is willing to buy at the price p, in dollars per pair be:

$$D(p) = 150 \cdot e^{-0.04p}$$

a) Find the Expenditure function for the consumer (the amount the consumers are spending, which is found by multiplying the number of items purchased (*D(p)* by the the price per item (*p*)), ($R(p) = p \cdot D(p)$).

b) Find the ***rate of change*** of the expenditure function ($R'(p)$).

c) Find the price that maximizes the consumer expenditure.

Solution:

a) The equation for Expenditure ($R(p)$) = (number of items)•(price/item); and is likewise equal to the Demand ($D(p)$), multiplied by the price (p).

$$R(p) = p \cdot D(p)$$

$$R(p) = p \cdot (150 \cdot e^{-0.04p}) = 150p \cdot e^{-0.04p}$$

b) Next, we take the derivative of the above equation using the product rule:

$$R'(p) = \frac{d}{dp} R(p) = \frac{d}{dp}[150p \cdot e^{-0.04p}]$$

$$R'(p) = 150\left[e^{-0.04p} \cdot \frac{d}{dp}(p) + p \cdot \frac{d}{dp}(e^{-0.04p})\right]$$

$$R'(p) = 150[e^{-0.04p} + p \cdot (-0.04 \cdot e^{-0.04p})]$$

$$R'(p) = 150[e^{-0.04p} - 0.04p \cdot e^{-0.04p}]$$

$$R'(p) = 150e^{-0.04p} \cdot (1 - 0.04p)$$

Example Problem #1 – Cont'd

a) To find the extreme values of a function, we need to find the critical points, by setting the derivative equal to zero ($R'(p) = 0$).

$$R'(p) = 150e^{-0.04p} \cdot (1 - 0.04p) = 0$$

Since the exponential function has a range greater than 0, $e^{-0.04t} \neq 0$

$$(1 - 0.04p) = 0$$

$$0.04p = 1$$

$$p = \frac{1}{0.04} = 25$$

Therefore, the price is: ($p = \$25$)

Next, finding the second derivative, yields:

$$R''(p) = \frac{d}{dp}E'(p) = \frac{d}{dp}[150e^{-0.04p} \cdot (1 - 0.04p)]$$

$$R''(p) = 150 \cdot \left[(1 - 0.04p)\frac{d}{dp}(e^{-0.04p}) + e^{-0.04p}\frac{d}{dp}(1 - 0.04p)\right]$$

$$R''(p) = 150 \cdot [(1 - 0.04p)(-0.04 \cdot e^{-0.04p}) + e^{-0.04p} \cdot (-0.04)]$$

$$R''(p) = 150 \cdot [-0.04(-0.04 \cdot e^{-0.04p})p - 0.04 \cdot e^{-0.04p} - 0.04e^{-0.04p}]$$

$$R''(p) = 150 \cdot [0.0016 \cdot p \cdot e^{-0.04p} - 0.08e^{-0.04p}]$$

$$R''(p) = -150 \cdot (0.0016) \cdot e^{-0.04p} \cdot [50 - p]$$

$$R''(p) = -0.24 \cdot e^{-0.04p} \cdot [50 - p]$$

Substituting ($p = \$25$) into the second derivative:

$$R''(25) = -0.24 \cdot e^{-0.04(25)} \cdot [50 - 25] = -2.2073$$

Substituting p = 25 into the second derivative ($R''(25) < 0$), which means Expenditure is maximized when the price is \$25.

Relative Rates of Change

Think about a car that values $30,000 and a TV that values $400. What if we drop each price by $100?

For the car that wouldn't make so much of a difference, as it is in fact only a 0.3% drop in price, but for the TV, it's a 25% drop in price, so that makes a huge difference.

The 0.3% change in price to the car, might not affect the demand too much, but the 25% drop in price for the TV probably would.

Relative Rate of Change

A relative rate of change in a function, say ($RRC(f(x))$) represents the change in the function with respect to the function itself. It is a percent change in the function.

$$RRC\big(f(x)\big) = \frac{d}{dx}\ln f(x)$$

or

$$RRC\big(f(x)\big) = \frac{f'(x)}{f(x)}$$

The reason why you can also look at this RRC as the derivative of the natural logarithm function of $f(x)$ is because, if you notice that is the formula that we have used in the last section for the derivative of the composition of a natural log function and another function.

Using a natural log first can simplify some functions and hence their derivative, but we don't have to use it.

Example Problem #2

The home prices in Orange County, California have been steadily increasing according to the formula ($f(t)$) dollars, where (t) is years since 2010.

$$f(t) = 500{,}000e^{0.0071t^2}$$

Find the ***relative rate of change*** in the price in 2019.

Solution:

Method #1:

$$RRC\big(f(t)\big) = \frac{f'(t)}{f(t)} = \frac{\frac{d}{dt}\left[500{,}000e^{0.0071t^2}\right]}{500{,}000 \cdot e^{.0071t^2}}$$

$$RRC\big(f(t)\big) = \frac{f'(t)}{f(t)} = \frac{500{,}000 \cdot \frac{d}{dt}\left[e^{0.0071t^2}\right]}{500{,}000 \cdot e^{.0071t^2}}$$

$$RRC\big(f(t)\big) = \frac{f'(t)}{f(t)} = \frac{\big(0.0071 \cdot (2t)\big)e^{0.0071t^2}}{e^{.0071t^2}}$$

$$RRC\big(f(t)\big) = 0.0142t$$

In year 2019, [$t = (2019 - 2010) = 9\ years$], the relative rate of change will be

$$RRC\big(f(9)\big) = 0.0142(9) = 0.1278$$, i.e. 12.78%

Example Problem #2 – Cont'd

Solution:

Method #2:

If we use the natural log version of the formula above, we can use the properties of logarithms first before we take the derivative.

Using the logarithmic rule: $[\ln(AB) = \ln(A) + \ln(B)]$

$$RRC\big(f(t)\big) = \frac{d}{dt}\ln f(t) = \frac{d}{dt}\left[\ln\left(500{,}000e^{0.0071t^2}\right)\right]$$

$$RRC\big(f(t)\big) = \frac{d}{dt}\left[\ln(500{,}000) + \ln\left(e^{0.0071t^2}\right)\right]$$

$$RRC\big(f(t)\big) = \frac{d}{dt}[\ln(500{,}000) + 0.0071t^2 \cdot \ln(e)]$$

$$RRC\big(f(t)\big) = \frac{d}{dt}[\ln(500{,}000)] + \frac{d}{dt}[0.0071t^2]$$

$$RRC\big(f(t)\big) = \left(\frac{1}{500{,}000}\right) \cdot \frac{d}{dt}(500{,}000) + 0.0071 \cdot (2t)$$

$$RRC\big(f(t)\big) = 0 + 2(0.0017t) = 0.0142t$$

$$RRC\big(f(t)\big) = 0.0142t$$

In year 2019, $[t = (2019 - 2010) = 9\ years]$, the relative rate of change will be

$$RRC\big(f(9)\big) = 0.0142(9) = 0.1278$$, i.e. 12.78%

Example Problem #3

A colony of mosquitoes has an initial population of 1000. After (t) days, the population is given by $(A(t))$.

$$A(t) = 1000e^{0.3t}$$

Show that the ***relative rate of change*** of the population, $(A'(t))$, to the population, $(A(t))$ is constant.

Solution:

First find $(A'(t))$. By using the chain rule, we have:

$$A'(t) = \frac{d}{dt}(1000e^{0.3t}) = 1000 \cdot \frac{d}{dt}(e^{0.3t})$$

$$A'(t) = 1000[0.3 \cdot e^{0.3t}]$$

$$A'(t) = 300e^{0.3t}$$

Thus, the relative rate of change of the population is given by:

$$RRC\big(A(t)\big) = \frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3$$

The ratio of the rate of change of the population to the population is the constant 0.3, or 30%.

As you notice, there is no variable (t) left in the expression, so it doesn't matter what the time is, the relative rate of change stays the same.

Elasticity of Demand

Elasticity of demand is a measure of how demand reacts to price changes.

It's normalized – that means the particular prices and quantities don't matter, and everything is treated as a percent change.

The formula for elasticity of demand involves a derivative, which is why we're discussing it here.

Elasticity of demand compares how the consumers demand compares to the changes in price.

Product, Good, or Service	Elasticity %	Demand	Elasticity and Revenue
Fresh Produce	3.08	Elastic – E > 1	Lower Prices to increase Revenue
Cars	1.89	Elastic – E > 1	Lower Prices to increase Revenue
Dining Out Meals	2.25	Elastic – E > 1	Lower Prices to increase Revenue
Rent/Housing	0.32	Inelastic – E < 1	Raise Prices to increase Revenue
Automotive Gas	0.45	Inelastic – E < 1	Raise Prices to increase Revenue
Clothing & Apparel	0.21	Inelastic – E < 1	Raise Prices to increase Revenue

The Elasticity of Demand is a ratio between percentage, hence it has no units.

The Elasticity of Demand is simply a number, which makes it easy to compare between different products or manufacturers.

Elasticity of Demand

Given a demand function that gives quantity in terms of (p)

$$E(p) = -\frac{Relative\ change\ in\ demand}{Relative\ change\ in\ price}$$

$$E(p) \quad = \quad -\frac{\frac{D'(p)}{D(p)}}{\frac{d}{dp}\ln p} \quad = \quad -\frac{\frac{D'(p)}{D(p)}}{\frac{1}{p}}$$

$$E(p) \quad = \quad -\frac{p \cdot D'(p)}{D(p)}$$

(Note that since demand is a decreasing function of (p), the derivative is negative.

That's why we are adding the negative sign, so ($E(p)$) will always be positive.)

- If **E < 1,** we say demand is **inelastic.**
  - In this case, "**Raise the price**" raising prices increases revenue.
- If **E > 1**, we say demand is **elastic.**
  - In this case, "**Lower the price**", lowering prices increases revenue.
- If **E = 1**, we say demand is **unitary**.
  - E = 1 at critical points of the revenue function.

Interpretation of elasticity:

If the price increases by 1%, the demand will decrease by **E**%.

Example Problem #4

A company sells (q) ribbon winders per year at \$$p$ per ribbon winder. The demand function for ribbon winders is given by ($D(p)$).

$$q = D(p) = 15000 - 50\,p$$

Find the elasticity of demand when the price is \$70 a piece? Will an increase in price lead to an increase in revenue?

Solution:

First, we need to take the derivative of the demand function ($q' = D'(p)$)

$$D'(p) = \frac{d}{dp}D(p) = \frac{d}{dp}(15000 - 50\,p) = -50$$

Next, substitute the appropriate parts to the Elasticity of Demand equation:

$$E(p) = -\frac{p \cdot D'(p)}{D(p)} = \frac{(-p)\cdot(-50)}{15000 - 50p} = \frac{50p}{15000 - 50p}$$

$$E(p) = \frac{p}{300 - p}$$

Next, substitute the price ($p = \$70$), into the Elasticity of Demand equation above:

$$E(70) = \frac{70}{300 - 70} = 0.3 < 1$$

E < 1, so demand is inelastic. Increasing the price by 1% would only cause a 0.3% drop in demand.

Increasing the price would lead to an increase in revenue, so it seems that the **company should increase its price**.

The demand for products that people have to buy, such as onions, tends to be inelastic. Even if the price goes up, people still have to buy about the same amount of onions, and revenue will not go down.

The demand for products that people can do without, or put off buying, such as cars, tends to be elastic. If the price goes up, people will just not buy cars right now, and revenue will drop.

Example Problem #5

A company finds the Demand $(D(p))$, in thousands, for their kites to be at a price of (p) dollars.

$$D(p) = 400 - p^2$$

Find the **elasticity of demand** when the price is $5, and when the price is $15?

Then find the price (p) that will maximize revenue?

Solution:

Calculating the derivative of the demand function first:

$$D'(p) = \frac{d}{dp}D(p) = \frac{d}{dp}(400 - p^2) = -2p$$

Now let's calculate the elasticity of demand:

$$E(p) = -\frac{p \cdot D'(p)}{D(p)}$$

$$E(p) = -\frac{p \cdot (-2p)}{400 - p^2} = \frac{2p^2}{400 - p^2}$$

$$E(5) = \frac{50}{400 - 25} = 0.13 < 1$$

.

The **demand is inelastic** when the price is $5. **Revenue could be raised by increasing prices**.

At a price of $5, a 1% increase in price, would **decrease demand** by 0.133%.

$$E(15) = \frac{450}{400 - 225} = 2.571 > 1.$$

The **demand is elastic** when the price is $15. **Revenue could be raised by decreasing prices**.

At a price of $15, a 1% increase in price, would **decrease demand** by 2.571%.

Example Problem #5 – Cont'd

To maximize the revenue, we could solve for when ($E(p) = 1$)

$$\frac{2p^2}{400 - p^2} = 1$$

$$2p^2 = 400 - p^2$$

$$p^2 = \frac{400}{3}$$

$$p = \sqrt{\frac{400}{3}} \approx 11.55$$

A price (p) of $11.55 will maximize revenue.

3.4 - EXERCISES

Find the relative rate of change of the following functions and evaluate them at the given values.			
1.	$f(x) = x^2 + 1$ $at\ x = 5$	2.	$f(x) = 3500e^{0.007x}$ $at\ x = 10$
3.	$f(x) = 5\sqrt{x-5}$ $at\ x = 9$	4.	$f(p) = 550pe^{p^2}$ $at\ p = 1$

Find the elasticity of demand of the following functions and determine if the demand is elastic, inelastic or unit elastic at the given number.			
5.	$D(p) = 500e^{-0.05p}$ $p = 100$	6.	$D(p) = 500 - 10p$ $p = 15$
7.	$D(p) = \frac{500}{p}$ $p = 50$	8.	$D(p) = \sqrt{25 - p^2}$ $p = 4$

9.	The demand of a certain item depends on the price charged for the item. If the demand is $D(p)$, many units for a price p, in dollars, then $$D(p) = 200e^{-0.04p}$$ a) Find the consumer expenditure. b) Find the price that maximizes consumer expenditure.
10.	The demand function for Alicia's oven mitts is given by $D(p)$ where p is the price in dollars. $$D(p) = -8p^2 + 750$$ Find the elasticity of demand when p = \$7.50. Will revenue increase if Alicia raises her price from \$7.50?
11.	The demand function for Shaki's danglies is given by $D(p)$ where p is the price in dollars per dangly. $$D(p) = -35p + 205$$ Find the elasticity of demand when $p = \$2.80$. Should Shaki raise or lower his price to increase revenue? Find the price that would maximize Shaki's revenue.
12.	The demand function for a certain textbook is given by $D(p)$. $$D(p) = \frac{800}{p^2}$$ Should the publisher raise or lower the price from the current price of \$160?

13.	A movie theater is planning to raise prices for the movie tickets from the current price of \$12. The demand function is $D(p)$. $$D(p) = 100\sqrt[3]{15-p}$$ Will this be a good idea?
14.	A country estimates that the demand for a certain make of car it exports is $D(p)$. $$D(p) = 1000e^{-0.38p}$$ Should it lower or raise the current price of \$20,000 it charges per car?

Solutions:

1. 0.3846 (or 38.46%)

2. 0.007

3. 0.125

4. 3

5. $E(100) = 5$, elastic

6. $E(15) = 3/7$, inelastic

7. $E(50) = 1$, unit-elastic

8. $E(4) = 16/9$, elastic

9. a) $E(p) = 200pe^{(-0.04p)}$, b) $25

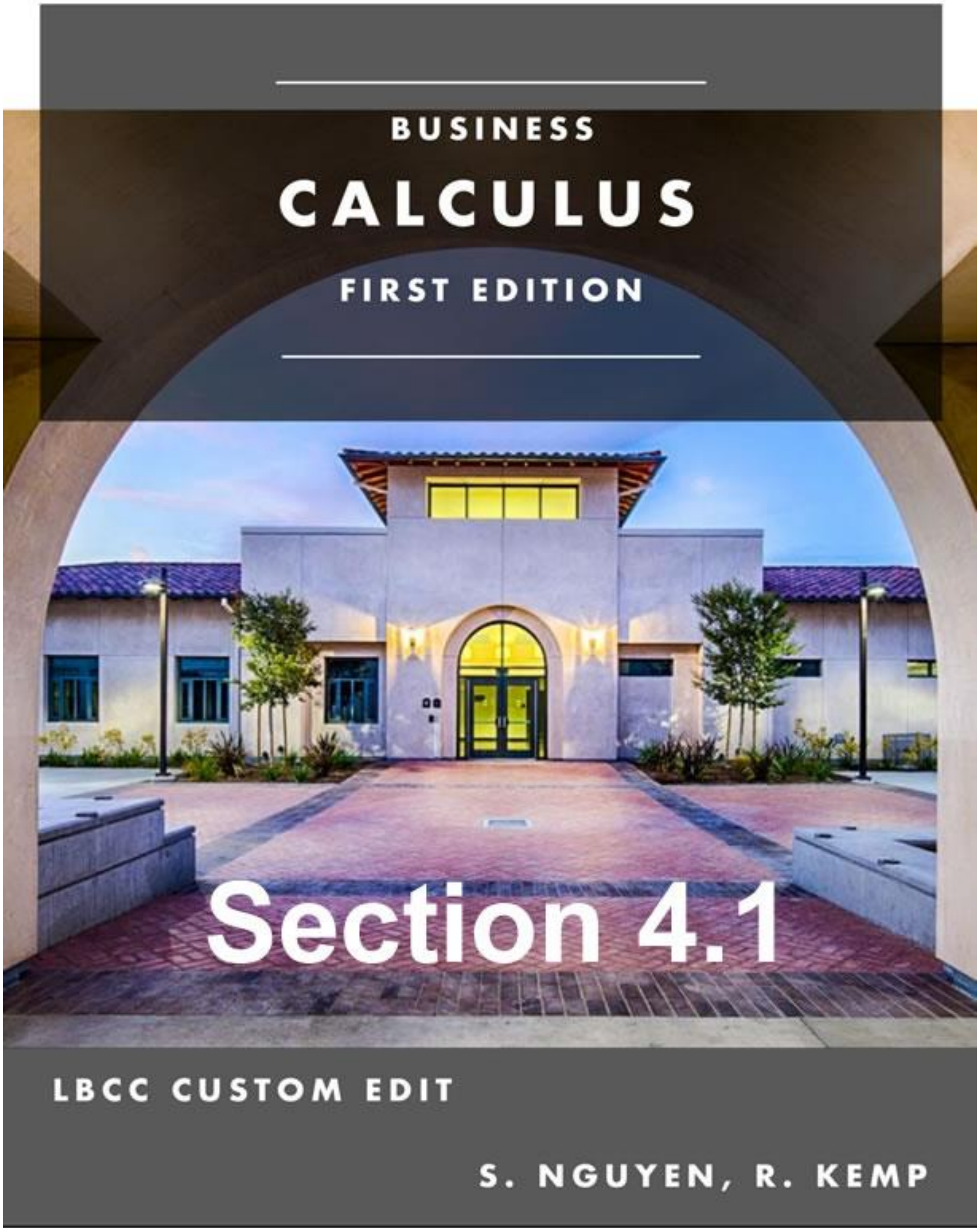
10. $E(7.50) = 3$, lower prices

11. $E(2.80) = 0.916$, increase prices. $2.93

12. $E(160) = 2$, lower prices

13.
$$E(P) = \frac{P}{3(15-P)} \qquad E(12) = \frac{12}{9} = 1.33 > 1$$
Elastic, Lower the Price

14.
$$E(P) = 0.38 \cdot P \qquad E(20000) = 7600 > 1$$
Elastic, Lower the Price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4.1 - PARTIAL DERIVATIVES - FUNCTIONS OF MULTI-VARIABLES

Introduction

Real-life is rarely as simple as one input – one output. Many relationships depend on lots of variables. Examples:

- If you put a principle deposit into an interest-bearing account, and let it sit, the amount you would have at the end of (3 years) depends on:
    - The Initial "Principle" Deposit - (P)
    - The annual interest rate - (r)
    - The number of compounding per year - (n).

$$f(P, r, n)$$

- The air resistance on a wing in a wind tunnel depends on:
    - the shape of the wing - (s)
    - the speed of the wind - (v)
    - the wing's orientation (pitch, yaw, and roll) - (w)
    - plus, a myriad of other things that I can't begin to describe.

$$f(s, v, w, \dots)$$

- The amount of your television cable bill depends on:
    - basic rate structure you have chosen - (b)
    - how many pay-per-view movies you ordered - (p)

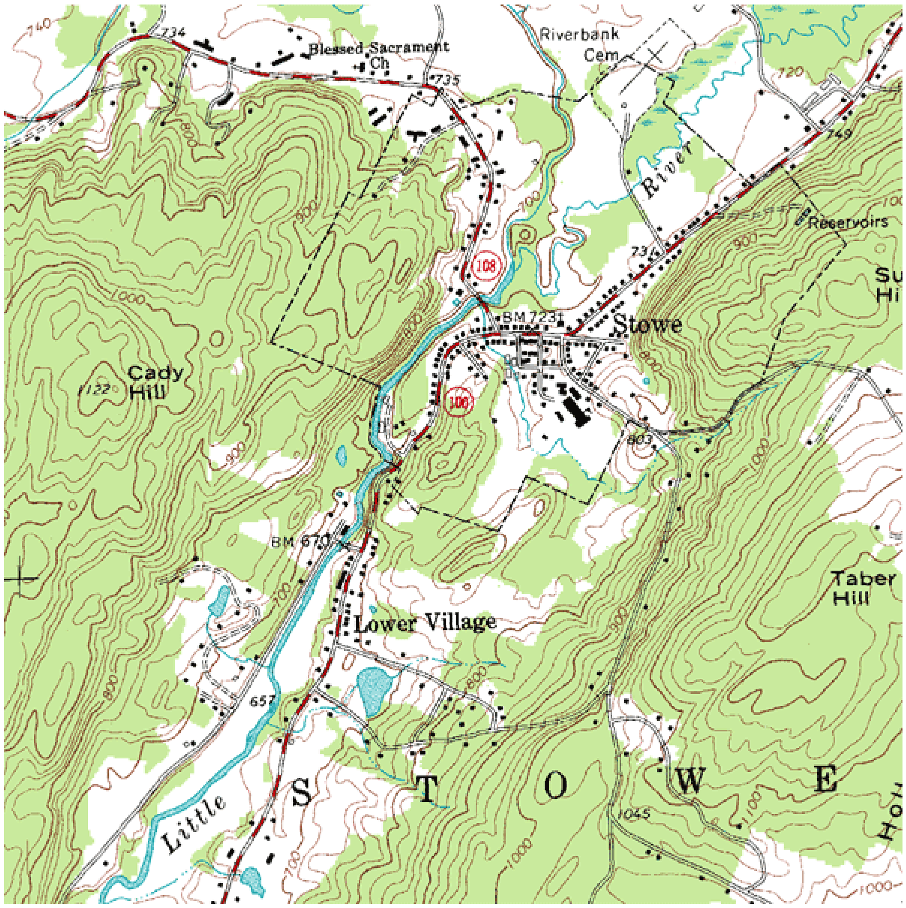
$$f(b, p)$$

Since the real world is so complicated, we want to extend our calculus ideas to functions of several variables.

Functions of several variables can be described numerically (a table), graphically, algebraically (a formula), or in words.

Topological Maps are Two Variable Data Plots

If you've ever hiked, you have probably seen a topographical map. Here is part of a topographic map of Stowe, Vermont.



(Figure courtesy of United States Geological Survey and http://en.wikipedia.org/wiki/File:Topographic_map_example.png.)

Points with the same elevation, are connected, with curves, so you can read not only your east-west and your north-south location, but also your elevation.

You may have also seen weather maps that use the same principle – points with the same temperature, are connected, with curves (isotherms), or points with the same atmospheric pressure, are connected, with curves (isobars).

These maps let you read not only a places location but also its temperature or atmospheric pressure.

In this section, we will use that same idea to make graphs of functions of two variables.

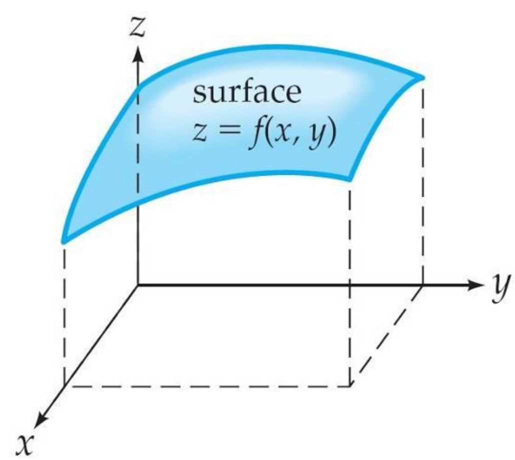
Functions of Two Variables

A function (f) that depends on one (1) variable, (x), is written ($f(x)$).

A function (f) that depends on two (2) variables, (x) and (y), is written ($f(x, y)$).

The "**Domain**" of the function (f), is the set of all ordered pairs (x, y) for which ($f(x, y)$) is defined.

The "**Range**" is the set of all resulting values of the function ($z = f(x, y)$).



A Function of Two Variables

A function (f) of two variables is a function, that is, to each **input** (x, y) is associated exactly one **output** ($f(x, y)$).

The **inputs** are ordered pairs: (x, y)

The **outputs** are single real numbers. $f(x, y)$

The **domain** of a function is the set of all possible inputs (ordered pairs): (x, y)

The **range** is the set of all possible outputs (real numbers): $f(x, y)$

The graph of a **two-variable function** is a **surface** whose height above the point in the ((x, y)) plane is: $z = f(x, y)$

Example Problem #1

Finding a Company's Cost Function in Two Variables

A company manufactures two products: An **Electric Skateboard** and an **Electric Bicycle**. The fixed cost is $2200.

It cost $120 to make each **Electric Skateboard**

It cost $180 to make each **Electric Bicycle**

Find the Cost function ($C(x, y)$) of producing (25) **Electric Skateboards** and (36) **Electric Bicycles**.

Solution:

Let:

The number of **Electric Skateboards** $= (x = 25)$

The number of **Electric Bicycles** $= (y = 36)$

The Company's Cost function ($C(x, y)$) is given:

$$C(x, y) = 120x + 180y + 2200$$

Next evaluate the Company's Cost function ($C(25, 36)$):

$$C(25, 36) = 120(25) + 180(36) + 2200$$

$$C(25, 36) = 3000 + 6480 + 2200 = 11680$$

$$C(25, 36) = \$11{,}680$$

Therefore, the **Cost** of producing (25) **Electric Skateboards** and (36) **Electric Bicycles**, is $11,680.

Cobb-Douglas – A Function of Two Variables

A **Production Function** is a function we use to model the output of a company. The one found most in business applications is called the "**Cobb-Douglas**" production function ($P(L, K)$)

The "Cobb-Douglas" production function ($P(L, K)$) expresses units of "Production" (P) as a function, of "Labor" (L) Units and "Capital" (K) units.

$$P(L, K) = \alpha L^{\beta} K^{1-\beta}$$

In this formula, (α) and (β) are given constants values. α is a positive value and β is a value between 0 and 1.

"Labor" is measured in "work-hours", i.e. the numbers of hours a company is emplying and paying its employees, and the "Capital" is measured in units of dollars invested in machinery, buildings, equipment, materials, utilities and so on.

Example Problem #2

Cobb-Douglas Production Function Evaluation

The "Cobb-Douglas" production function ($P(L, K)$) predicts the output of a company. Find $P(120, 260)$?

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

Solution:

$$P(L, K) = 1.01(120)^{0.75}(260)^{0.25}$$

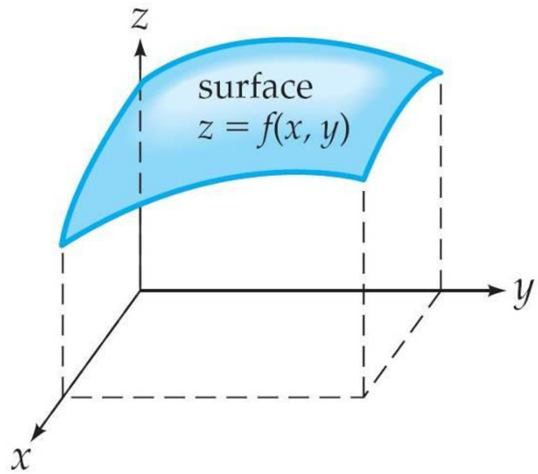
$$P(L, K) \cong 147$$

Interpretation:
If 120 units of Labor and 260 units of Capital have been invested, the comany will produce approximately 147 units of what they are producing.

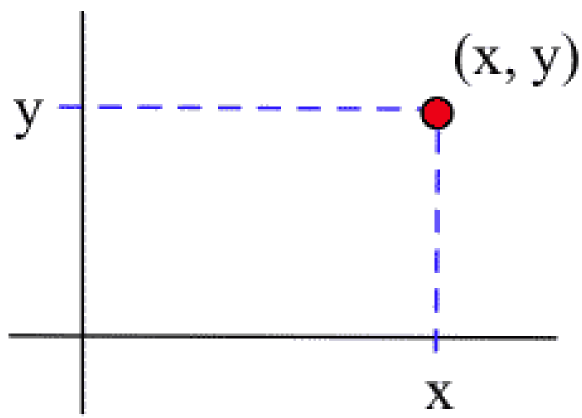
You can read more about Cobb-Douglas Production functions at: http://en.wikipedia.org/wiki/Cobb-Douglas.
You can read about other kinds of production functions at: http://en.wikipedia.org/wiki/Production_function.

Graphs

The graph of a function of two variables is a surface in three-dimensional space, defined by a 3-dimensional rectangular coordinate system.



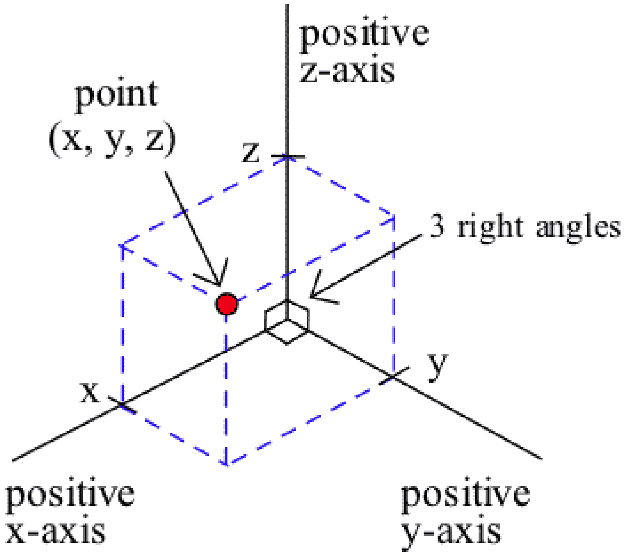
In the 2-dimensional rectangular coordinate system we have two coordinate axes that meet at right angles at the origin, and it takes two numbers, or an ordered pair (x, y), to specify the rectangular coordinate location of a point in the two 2-dimensional plane.



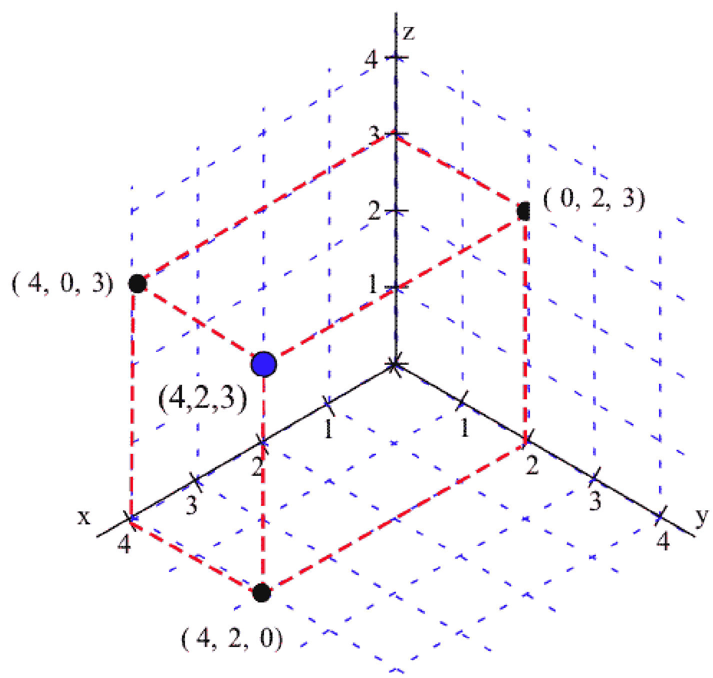
Each ordered pair (x, y) specifies the location of exactly one point, and the location of each point is given by exactly one ordered pair (x, y).

The (x) and (y) values are the coordinates of the point (x, y).

The situation in three 3-dimensions is very similar. In the 3-dimensional rectangular coordinate system we have three coordinate axes that meet at right angles, and three numbers, an ordered triple (x, y, z), are needed to specify the location of a point.



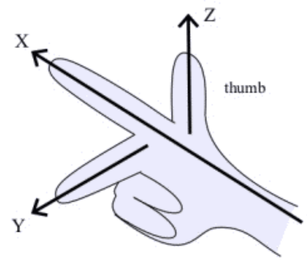
Each ordered triple (x, y, z) specifies the location of exactly one point, in a three 3-dimensional coordinate system. The (x), (y), and (z) values are the coordinates of the point (x, y, z). The figure below shows the location of the point (4, 2, 3).



Locating the point (4, 2, 3)

Typically, we use a right-hand orientation. To see what this means, imagine your right hand in front hand in front of you with the palm toward your face, your thumb pointing up, you index finger straight out, and your next finger toward your face (and the two bottom fingers bent into the palm).

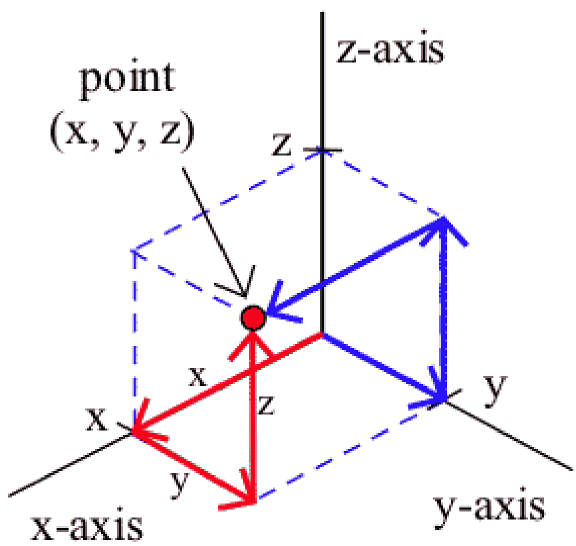
In the right-hand coordinate system, your thumb points along the positive z-axis, your index finger along the positive x-axis, and the other finger along the positive y-axis.



Right-hand coordinate system

Other orientations of the axes are possible and valid (with appropriate labeling), but the right-hand system is the most common orientation and is the one we will generally use. If another orientation is used, then the axes will be explicitly labeled.

Each ordered triple (x, y, z) specifies the location of a single point, and this location point can be plotted by locating the point $(x, y, 0)$ in the (xy-plane) and then going up (z) units (the red path in the figure below).

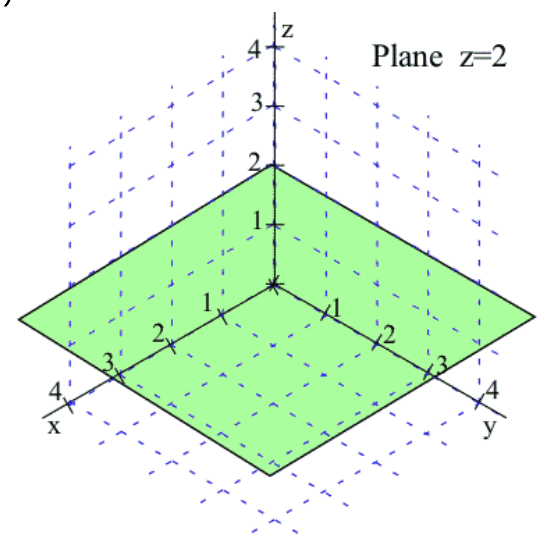


We could also get to the same (x, y, z) point in other ways. For instance, we could start by finding the point $(x, 0, z)$ in the (xz-plane) and then going (y) units parallel to the y-axis. Or by finding $(0, 0, z)$ in the (xz on the (yz-plane) and then going (x) units parallel to the x-axis (the blue path in the figure above).

Once we can locate points, we can begin to consider the graphs of various collections of points.

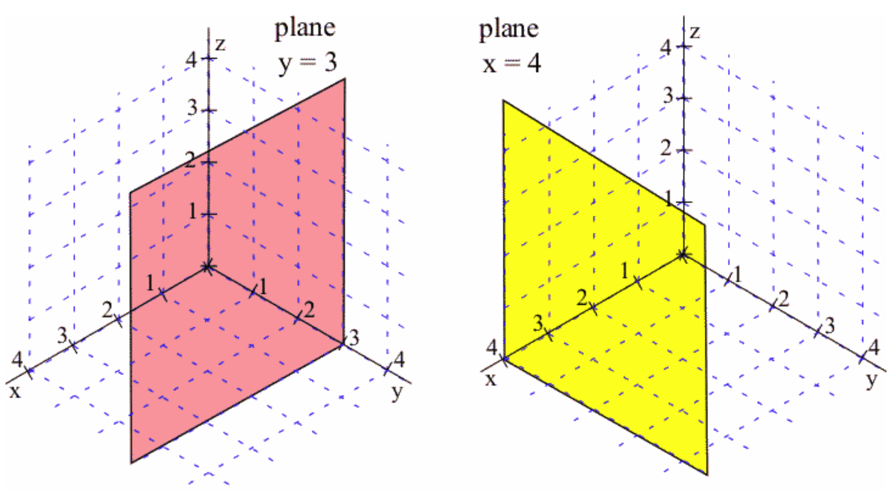
By the graph of "($z = 2$)" we mean the collection of all points (x, y, z) which have the form "$(x, y, 2)$".

Since no condition is imposed on the (x) and (y) variables, they take all possible values. The graph of "($z = 2$)" is a plane parallel to the (xy-plane) and 2 units above the (xy-plane).



Similarly, the graph below, of "($y = 3$)" is a plane parallel to the (xz-plane).

And the graph below of "($x = 4$)" is a is a plane parallel to the (yz-plane).



Planes y = 3 and x = 4

(Note: The planes have been drawn as rectangles, but they extend infinitely far.)

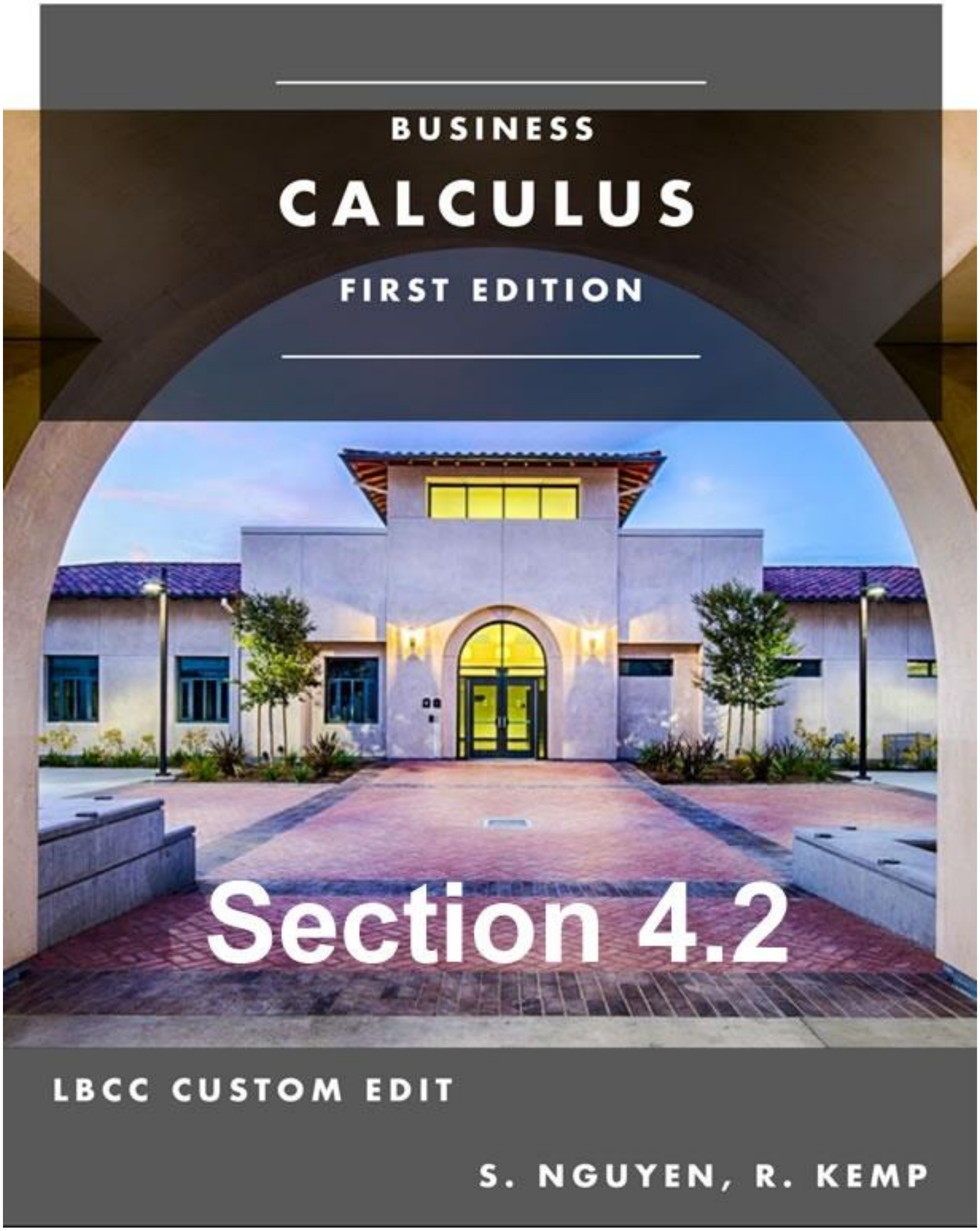
4.1 - EXERCISES

For each function, evaluate the given functions.	
1.	$f(x,y) = x^2 + 3xy + 2y^2 - 5x + 3y$ Find $f(4,2)$
2.	$f(x,y) = \sqrt{3xy - x^2y + 20y}$ Find $f(0,5)$
3.	$f(x,y) = e^{(y^2 + xy - 6)}$ Find $f(1,-3)$
4.	$f(x,y) = \dfrac{4x^2 + 5y^2}{xy}$ Find $f(-1,1)$
5.	$f(x,y) = \ln(2xy - 3x - 1)$ Find $f(2,2)$
6.	$f(x,y) = e^{(lnx)} + \ln(y^3) - x^{10}$ Find $f(1,e)$

7.	$f(x,y,z) = xyz^2 \cdot ln\sqrt{xy}$ Find $f(-1,-1,4)$
8.	$g(x,y,z) = [xe^{2y} - ye^{2z} - ze^{2x}]$ Find $g(1,-1,1)$
9.	The "Cobb-Douglas" production function ($P(L,K)$) predicts the output of the American Economy by the function. $P(L,K) = 4L^{0.8}K^{0.2}$ Find $P(300,200)$?
10.	The "Cobb-Douglas" production function ($P(L,K)$) predicts the output of a cell phone company, by the function. $P(L,K) = 3L^{\frac{1}{2}}K^{\frac{1}{2}}$ Find $P(600,1200)$?
11.	The maximum duration of the climb of an Air Balloon ($D(x,y)$) in minutes, where (x) is the density of the air, and the height (y) of the climb is given by. $D(x,y) = \frac{42x}{y + 42}$ Find $D(80,42)$?
12.	The cost function for a company, in dollars, that produces airbags for trucks and SUV's is $C(x,y) = x^2 + y^3 - 50xy + 1000$ where x is the number of truck airbags and y is the number of SUV airbags produced. Evaluate C (100, 70) and interpret the answer.

Solutions

1. 34
2. 10
3. 1
4. -9
5. 0
6. 3
7. 0
8. $e^{-2} = 0.135335$
9. $1106.53 \approx 1107$ units
10. $P = 2545.548 \approx 2546$ units
11. $40\ minutes$
12. The cost of producing 100 truck airbags and 70 SUV airbags is $4000.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4.2 - PARTIAL DERIVATIVES - FUNCTIONS OF TWO VARIABLES

Introduction

Derivatives tell us about the shape of the graph of a function, and let us find local max and mins. We want to be able to do the same thing with a function of two variables.

First let's think. Imagine a surface, the graph of a function of two variables. Imagine that the surface is smooth and has some hills and some valleys. Concentrate on one point on your surface.

What do we want the derivative to tell us? It ought to tell us how quickly the height of the surface changes as we move…

Wait, which direction do we want to move? This is the reason that derivatives are more complicated for functions of several variables – there are so many (in fact, infinitely many) directions we could move from any point.

It turns out that our idea of fixing one variable and watching what happens to the function as the other variable changes is the key to extending the idea of derivatives to more than one variable.

Functions of several variables have several derivatives, called "**Partial Derivatives**", one for each variable.

Partial Derivatives

The "Partial Derivatives" are "Rates of Change". Since partial derivatives are just "ordinary" derivatives with other variables held constant, they give "instantaneous rates of change" with respect to one variable at a time.

They are called "Partial Derivatives" because not all variables are changed at once. When taking a derivative of a function of several variables, only a "partial" of the function changes during a derivative.

Suppose that a function of two variables is given by:

$$z = f(x, y)$$

The partial derivative of a function (f) with respect to (x) is the derivative of the function ($z = f(x, y)$), where we think of (x) as the only variable, while (y) is held constant.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y)$$

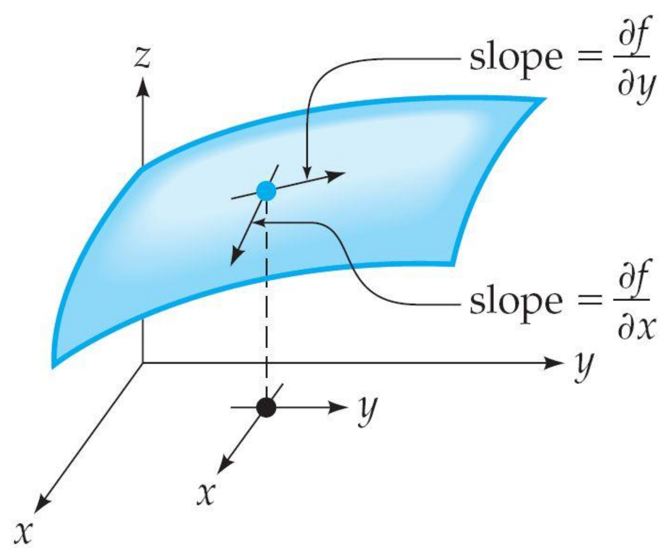
The partial derivative of the function (f) with respect to (y) is the derivative of the function ($z = f(x, y)$) where we think of (y) as the only variable, while (x) is held constant.

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y)$$

Geometrically

Geometrically the partial derivative with respect to (x) gives the slope of the curve as you travel along a cross-section, a curve on the surface parallel to the x-axis.

The partial derivative with respect to (y) gives the slope of the cross-section parallel to the y-axis.



Partial derivatives are slopes.

A function ($z = f(x,y)$) represents a surface in three-dimensional space, and the partial derivatives are the slopes along the surface in different directions:

- the partial ($\frac{\partial}{\partial x} f(x,y)$) gives the slope of the surface "**in the x-direction**,"; at the point (x, y).
- the partial ($\frac{\partial}{\partial y} f(x,y)$) gives the slope of the surface "**in the y-direction**" at the point (x, y).

A function ($z = f(x,y)$) has two "Partial Derivatives", one respect to (x), and the other with respect to (y).

Partial Derivatives

The partial derivative of a function ($z = f(x,y)$) with respect to (x):

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

The partial derivative of a function ($z = f(x,y)$) with respect to (y):

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Provided, that the limits exist.

The "Partial Derivatives" are written with a cursive stylized d, ∂ , derivative ($\frac{\partial}{\partial y}$), instead of ($\frac{dz}{dy}$), and are often called Partials.

Therefore, the "Partial Derivative" with respect to (x) is just the “Derivative” with respect to (x), with (y) held constant.

$$f_x(x,y) \;=\; \frac{\partial}{\partial x}f(x,y) \;=\; \begin{bmatrix} Derivative\ of\ (f)\ with\ respect \\ to\ (x), with\ (y)\ held\ constant \end{bmatrix}$$

Therefore, the “Partial Derivative” with respect to (y) is just the “Derivative” with respect to (y), with (x) held constant.

$$f_y(x,y) \;=\; \frac{\partial}{\partial y}f(x,y) \;=\; \begin{bmatrix} Derivative\ of\ (f)\ with\ respect \\ to\ (y), with\ (x)\ held\ constant \end{bmatrix}$$

The idea of a partial derivative works perfectly well for a function of several variables.

The main idea is that you focus on one variable, where you take the derivative, and act as if all the other variables are constants.

Basic Rules of Partial Derivatives

Let’s review a few basic calculus rules that will governing derivatives and constants.

- The derivative of a constant (C) is zero.

$$\frac{d}{dx}C \;=\; \frac{\partial}{\partial x}C \;=\; 0$$

- The constant multiplying function – all constants are moved forward.

$$\frac{d}{dx}(Cx^4) \;=\; \frac{\partial}{\partial x}(Cx^4) \;=\; C\frac{\partial}{\partial x}(x^4) \;=\; C\cdot(4x^3) \;=\; 4C(x^3)$$

Example Problem #1

Find the Partial Derivatives with Respect to (x) and with respect to (y).

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^4y^5)$$

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4y^5)$$

Solution:

The notation ($\frac{\partial}{\partial x}$) means to differentiate with respect to (x), while holding (y), and therefore (y^5) is constant.

We therefore differentiate the (x^4) and carry the constant along, moving it to the front of the equation.

$$f_x(x,y) = \frac{\partial z}{\partial x} = y^5\frac{\partial}{\partial x}(x^4) = y^5(4x^3) = 4x^3y^5$$

The notation ($\frac{\partial}{\partial y}$) means to differentiate with respect to (y), while holding (x), and therefore (x^4) is constant.

We therefore differentiate the (y^5) and carry the constant along, moving it to the front of the equation.

$$f_y(x,y) = \frac{\partial z}{\partial y} = x^4\frac{\partial}{\partial y}(y^5) = x^4(5y^4) = 5x^4y^4$$

Example Problem #2

Find the Partial Derivatives

$$z = f(x, y) = (y^8 + 30)$$

$$f_x(x, y) = \frac{\partial}{\partial x}(y^8 + 30) \quad ; \quad f_y(x, y) = \frac{\partial}{\partial y}(y^8 + 30)$$

Solution:

Notice, that in the partial $(\frac{\partial}{\partial x})$ there is no (x) variable to differentiate; and the (y) and the (30) are both constants in this case.

However, there is a derivative that emerges when there is a partial with respect to $(\frac{\partial}{\partial y})$.

$$\frac{\partial}{\partial x}(y^8 + 30) = 0 \quad ; \quad \frac{\partial}{\partial y}(y^8 + 30) = 8y^7$$

Example Problem #3

Find the Partial Derivatives

$$z = f(x, y) = x^3$$

$$f_x(x, y) = \frac{\partial}{\partial x}(x^3) \quad ; \quad f_y(x, y) = \frac{\partial}{\partial y}(x^3)$$

Solution:

Notice, that in the partial $(\frac{\partial}{\partial y})$ there is no (y) variable to differentiate; and the (x^3) is a constant in this case.

However, there is a derivative that emerges when there is a partial with respect to $(\frac{\partial}{\partial x})$.

$$\frac{\partial}{\partial x}(x^3) = 3x^2 \quad ; \quad \frac{\partial}{\partial y}(x^3) = 0$$

Example Problem #4

Find the Partial Derivatives of a Polynomial with Two Variables

$$f(x,y) = (3x^5 - 2x^2y^3 - y^4 + 5x + 8)$$

Solution:

Notice, that here (y) is constant.

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(3x^5 - 2x^2y^3 - y^4 + 5x + 8)$$

$$f_x(x,y) = 3\frac{\partial}{\partial x}(x^5) - 2y^3\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(y^4) + 5\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(8)$$

$$f_x(x,y) = 3(5x^4) - 2y^3(2x) - 0 + 5(1) + 0$$

$$f_x(x,y) = [15x^4 - 4xy^3 + 5]$$

Notice, that here (x) is constant

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(3x^5 - 2x^2y^3 - y^4 + 5x + 8)$$

$$f_y(x,y) = \frac{\partial}{\partial y}(3x^5) - 2x^2\frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(y^4) + 5\frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(8)$$

$$f_y(x,y) = 0 - 2x^2(3y^2) - 4y^3 + 0 + 0$$

$$f_y(x,y) = [-6x^2y^2 - 4y^3]$$

Example Problem #5

Find the Partial Derivatives – ($f_x(x,y)$) & ($f_y(x,y)$)

$$f(x,y) \quad = \quad (3x^2 + 5y^3 - 10xy)^5$$

Solution:

Notice that in this case we will have to use the chain rule to take the partial derivatives, as our polynomial function is wrapped by a power function. We must first take the derivative of the outside function, times the derivative of the inside function.

Notice, that here (y) is constant

$$f_x(x,y) \quad = \quad \frac{\partial}{\partial x} f(x,y) \quad = \quad \frac{\partial}{\partial x}(3x^2 + 5y^3 - 10xy)^5$$

$$f_x(x,y) \quad = \quad 5(3x^2 + 5y^3 - 10xy)^{(5-1)} \left[\frac{\partial}{\partial x}(3x^2 + 5y^3 - 10xy)\right]$$

$$f_x(x,y) \quad = \quad 5(3x^2 + 5y^3 - 10xy)^4(6x - 10y)$$

Notice, that here (x) is constant.

$$f_y(x,y) \quad = \quad \frac{\partial}{\partial y} f(x,y) \quad = \quad \frac{\partial}{\partial y}(3x^2 + 5y^3 - 10xy)^5$$

$$f_y(x,y) \quad = \quad 5(3x^2 + 5y^3 - 10xy)^{(5-1)} \left[\frac{\partial}{\partial y}(3x^2 + 5y^3 - 10xy)\right]$$

$$f_y(x,y) \quad = \quad \frac{\partial}{\partial y} f(x,y) \quad = \quad 5(3x^2 + 5y^3 - 10xy)^4(15y^2 - 10x)$$

Example Problem #6

Find the Partial Derivatives – ($f_x(x,y)$) & ($f_y(x,y)$)

$$f(x,y) \;=\; e^{(2x\,-\,5y)}$$

Solution:

Notice, that here (y) is constant.

$$f_x(x,y) \;=\; \frac{\partial}{\partial x} f(x,y) \;=\; \frac{\partial}{\partial x}\left[e^{(2x\,-\,5y)}\right]$$

$$f_x(x,y) \;=\; e^{(2x\,-\,5y)} \cdot \frac{\partial}{\partial x}(2x \;-\; 5y)$$

$$f_x(x,y) \;=\; 2e^{(2x\,-\,5y)}$$

Notice, that here (x) is constant.

$$f_y(x,y) \;=\; \frac{\partial}{\partial y} f(x,y) \;=\; \frac{\partial}{\partial y}\left[e^{(2x\,-\,5y)}\right]$$

$$f_y(x,y) \;=\; e^{(2x\,-\,5y)} \cdot \frac{\partial}{\partial y}(2x \;-\; 5y)$$

$$f_y(x,y) \;=\; -5e^{(2x\,-\,5y)}$$

Now let us also not only find the partial derivatives, but also evaluate them at certain given input points (x, y).

Example Problem #7

Given the function

$$f(x,y) = (x^4y^2 + 3)^3$$

Find $f_x(1,2)$ and $f_y(1,2)$

Solution:

Find ($f_x(x,y)$), that here (y) is constant.

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(x^4y^2 + 3)^3$$

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = 3(x^4y^2 + 3)^2 \left[\frac{\partial}{\partial x}(x^4y^2 + 3)\right]$$

$$f_x(x,y) = 12x^3y^2 \cdot (x^4y^2 + 3)^2$$

Next, substitute the values ($f_x(1,2)$).

$$f_x(1,2) = 12(1)^3(2)^2 \cdot [(1)^4(2)^2 + 3]^2 = 48(7)^2 = (48)(49)$$

$$f_x(1,2) = 2352$$

Find ($f_y(x,y)$), that here (x) is constant.

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(x^4y^2 + 3)^3$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = 3(x^4y^2 + 3)^2 \left[\frac{\partial}{\partial y}(x^4y^2 + 3)\right]$$

$$f_y(x,y) = 6x^4y \cdot (x^4y^2 + 3)^2$$

Next, substitute the values ($f_y(1,2)$).

$$f_y(1,2) = 6(1)^4(2) \cdot [(1)^4(2)^2 + 3]^2 = 12(7)^2 = (12)(49)$$

$$f_y(1,2) = 588$$

Example Problem #8

Given the function

$$g(x, y) = ln(x^2 + xy + y^2)$$

Find $g_x(1,1)$ *and* $g_y(1,1)$.

Solution:

Find ($g_x(x, y)$), that here (y) is constant.

$$g_x(x, y) = \frac{\partial}{\partial x} g(x, y) = \frac{\frac{\partial}{\partial x}(x^2 + xy + y^2)}{(x^2 + xy + y^2)}$$

$$g_x(x, y) = \frac{2x + y}{(x^2 + xy + y^2)}$$

Next, substitute the values ($g_x(1,1)$).

$$g_x(1,1) = \frac{2(1) + 1}{((1)^2 + (1)(1) + (1)^2)} = \frac{3}{3} = 1$$

$$g_x(1,1) = 1$$

Find ($g_y(x, y)$), that here (x) is constant.

$$g_y(x, y) = \frac{\partial}{\partial y} g(x, y) = \frac{\frac{\partial}{\partial y}(x^2 + xy + y^2)}{(x^2 + xy + y^2)}$$

$$g_y(x, y) = \frac{\partial}{\partial y} g(x, y) = \frac{x + 2y}{(x^2 + xy + y^2)}$$

Next, substitute the values ($g_y(1,1)$).

$$g_y(1,1) = \frac{1 + 2(1)}{((1)^2 + (1)(1) + (1)^2)} = \frac{3}{3} = 1$$

$$g_y(1,1) = 1$$

Example Problem #9

Given the function

$$h(x,y) = \frac{2xy}{x^2 - y^2}$$

Find $h_x(0,2)$ $h_y(0,2)$

Solution:

Find ($h_x(x,y)$), that here (y) is constant.

$$h_x(x,y) = \frac{\partial}{\partial x}h(x,y) = \frac{2y(x^2 - y^2) - 2xy(2x)}{(x^2 - y^2)^2}$$

$$h_x(x,y) = \frac{\partial}{\partial x}h(x,y) = \frac{-2y^3 - 2x^2y}{(x^2 - y^2)^2}$$

Next, substitute the values ($h_x(0,2)$).

$$h_x(0,2) = \frac{-2(2)^3 - 0}{(0 - (2)^2)^2} = \frac{-16}{16} = -1$$

Find ($h_y(x,y)$), that here (x) is constant.

$$h_y(x,y) = \frac{\partial}{\partial y}h(x,y) = \frac{2x(x^2 - y^2) - 2xy(-2y)}{(x^2 - y^2)^2}$$

$$h_y(x,y) = \frac{\partial}{\partial y}h(x,y) = \frac{2x^3 + 2x^2y}{(x^2 - y^2)^2}$$

Next, substitute the values ($h_y(0,2)$).

$$h_y(0,2) = \frac{0 - (0)}{(0 - (2)^2)^2} = \frac{0}{16} = 0$$

Partial Derivatives in Three or More Variables

Not all functions (f) are limited to two variables. Some functions have three, four, five or more variables.

$$f(x, y, z, t) = 2x + 3y - 5z - 9t$$

The calculus of "Partial Derivatives" in three or more variables is defined the same way as the "Partial Derivatives" using two variables, i.e, the "Partial Derivative" of ($f(x, y, z, t)$) with respect to one variable is the derivative with respect to that variable, holding all other variables constant.

Example Problem #10

Find all Partial Derivatives of a Function in Three Variables

$$f(x, y, z) = 5x^2y^3z^4$$

Solution:

$$f_x(x, y, z) = \frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x}(5x^2y^3z^4) = 10xy^3z^4$$

$$f_y(x, y, z) = \frac{\partial}{\partial y} f(x, y, z) = \frac{\partial}{\partial y}(5x^2y^3z^4) = 15x^2y^2z^4$$

$$f_z(x, y, z) = \frac{\partial}{\partial z} f(x, y, z) = \frac{\partial}{\partial z}(5x^2y^3z^4) = 20x^2y^3z^3$$

Example Problem #11

Find all Partial Derivatives of a Function in Four Variables

$$f(x, y, z, t) = 35x^2t - \frac{1}{z} + yz^2$$

Solution:

$$f_x(x, y, z, t) = \frac{\partial}{\partial x}\left(35x^2t - \frac{1}{z} + yz^2\right) = 70xt$$

$$f_y(x, y, z, t) = \frac{\partial}{\partial y}\left(35x^2t - \frac{1}{z} + yz^2\right) = z^2$$

$$f_z(x, y, z, t) = \frac{\partial}{\partial z}\left(35x^2t - \frac{1}{z} + yz^2\right) = \frac{1}{z^2} + 2yz$$

$$f_t(x, y, z, t) = \frac{\partial}{\partial t}\left(35x^2t - \frac{1}{z} + yz^2\right) = 35x^2$$

Partials in Business and Economics

The calculus of “Partial Derivatives” also works for “Cost”, “Revenue”, and “Profit” functions. The partials give the “marginals” for one variable at time when the other variables are held constant.

That is the “Partial Derivative” of $(f(x, y))$ gives the marginals, but now we will apply the concept to a function with two different products; where one variable, changes while holding the other variable constant.

The use of partials to find changes in revenue, cost, and profit resulting from one additional unit is called “Marginal Analysis”.

Partial Derivatives as Marginal (Profit, Revenue, Cost) Functions

Let, $((f(x, y))$ be the total (**Profit, Revenue, Cost**) for (x) units of product (**A**) and (y) units of product (**B**).

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \begin{bmatrix} Marginal\ Function\ for\ product\ (A) \\ when\ production\ of\ product\ (B) \\ is\ held\ constant \end{bmatrix}$$

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \begin{bmatrix} Marginal\ Function\ for\ product\ (B) \\ when\ production\ of\ product\ (A) \\ is\ held\ constant \end{bmatrix}$$

Example Problem #12

A company's "Profit" from producing (x – stoves) and (y – refrigerators) per day is given by the function ($P(x, y)$). Evaluate: $P_x(16, 25)$ and $P_y(16, 25)$

Find the marginal profit functions.

$$P(x, y) = 12x^{\frac{3}{2}} + 8y^{\frac{3}{2}} + xy$$

Solution:

First, let's take the partial with respect to ($P_x(x, y)$)

$$P_x(x, y) = \frac{\partial}{\partial x}\left(12x^{\frac{3}{2}} + 8y^{\frac{3}{2}} + xy\right)$$

$$P_x(x, y) = \left[12\left(\frac{3}{2}\right)x^{\frac{1}{2}} + y\right] = \left[18x^{\frac{1}{2}} + y\right]$$

$$P_x(x, y) = \left[18\sqrt{x} + y\right]$$

Next, let's take the partial with respect to ($P_y(x, y)$)

$$P_y(x, y) = \frac{\partial}{\partial y}\left(12x^{\frac{3}{2}} + 8y^{\frac{3}{2}} + xy\right)$$

$$P_y(x, y) = \left[8\left(\frac{3}{2}\right)y^{\frac{1}{2}} + x\right] = \left[12y^{\frac{1}{2}} + x\right]$$

$$P_y(x, y) = \left[12\sqrt{y} + x\right]$$

Next, substitute the values ($P_x(16, 25)$) and ($P_y(16, 25)$).

$$P_x(16, 25) = \left[18\sqrt{16} + 25\right] = [18(4) + 25] = 97$$

$$P_y(16, 25) = \left[12\sqrt{25} + 16\right] = [12(5) + 16] = 76$$

The "Profit" increases by about (\$97.00) per additional "stove" and by about (\$76.00) per additional "refrigerator" when producing (16 stoves) and (25 refrigerator) per day.

Cobb-Douglas - Partials in Business and Economics

The "Cobb-Douglas" production function ($P(L, K)$) expresses units of "Production" (P) as a function of "Labor" (L) Units and "Capital" (K) units.

So, each of the partial derivatives will give us the rates of change in production with respect to one of the variables (L) or (K), while the other is held constant.

$$P(L, K) = aL^b K^{1-b}$$

Example Problem #13

Interpreting the Partial Derivatives of a Cobb-Douglas Production Function

Given $$P(L, K) = 16L^{0.4}K^{0.6}$$

Find and Interpret

$$P_L(140, 210) \quad \text{and} \quad P_K(140, 210)$$

Solution:

Find ($P_L(L, K)$), that here (K) is constant.

$$P_L(L, K) = \frac{\partial}{\partial L}P(L, K) = 16K^{0.6}\left[\frac{\partial}{\partial L}L^{0.4}\right] = 16K^{0.6}[(0.4)L^{-0.6}]$$

$$P_L(L, K) = \frac{\partial}{\partial L}P(L, K) = 6.4K^{0.6}L^{-0.6} = 6.4\left(\frac{K^{0.6}}{L^{0.6}}\right) = 6.4\left(\frac{K}{L}\right)^{0.6}$$

Find ($P_K(L, K)$), that here (K) is constant.

$$P_K(L, K) = \frac{\partial}{\partial K}P(L, K) = 16L^{0.4}\left[\frac{\partial}{\partial K}K^{0.6}\right] = 16L^{0.4}[(0.6)K^{-0.4}]$$

$$P_K(L, K) = \frac{\partial}{\partial K}P(L, K) = 9.6L^{0.4}K^{-0.4} = 9.6\left(\frac{L^{0.4}}{K^{0.4}}\right) = 9.6\left(\frac{L}{K}\right)^{0.4}$$

Example Problem #13 – Cont'd

Interpreting the Partial Derivatives of a Cobb-Douglas Production Function

Find and Interpret

$$P_L(140, 210) \quad \text{and} \quad P_K(140, 210)$$

$$P(L, K) = 16L^{0.4}K^{0.6}$$

Solution:

Next, substitute the values ($P_L(L, K) = P_L(140, 210)$).

$$P_L(140, 210) = 6.4\left(\frac{K}{L}\right)^{0.6}$$

$$P_L(140, 210) = 6.4\left(\frac{210}{140}\right)^{0.6} = 6.4\left(\frac{3}{2}\right)^{0.6}$$

$$P_L(140, 210) \cong 8.163 \ units$$

Next, substitute the values ($P_K(L, K) = P_K(140, 210)$).

$$P_K(140, 210) = 9.6\left(\frac{L}{K}\right)^{0.4}$$

$$P_K(140, 210) = 9.6\left(\frac{140}{210}\right)^{0.4} = 9.6\left(\frac{2}{3}\right)^{0.4}$$

$$P_K(140, 210) \cong 8.163 \ units$$

Interpretation: The "**Marginal Productivity of Labor**" is ($P_L = 8.163$), means that "Production" increases by about (8.16 units) for each additional unit of "**Labor**" when [L = 140 and K = 200].

Interpretation: The "**Marginal Productivity of Capital**" if ($P_K = 8.163$), means that "Production" increases by about (8.16 units) for each additional unit of "**Capital**" when [L = 140 and K = 200].

You can read more about Cobb-Douglas Production functions at http://en.wikipedia.org/wiki/Cobb-Douglas.

You can read about other kinds of production functions at http://en.wikipedia.org/wiki/Production_function.

4.2 - EXERCISES

For each function, find the partials derivatives. $a.\ f_x(x,y)$ $b.\ f_y(x,y)$			
1.	$f(x,y) = x^4 + 5x^2y^2 - 4y^2 + x - y$	2.	$f(x,y) = (2x + 2y)^3$
3.	$f(x,y) = 220x^{0.10}y^{0.20}$	4.	$f(x,y) = 20x^{\frac{1}{4}}y^{\frac{1}{5}} - 7$
5.	$f(x,y) = ln(x^4 + y^4)$	6.	$f(x,y) = e^{2xy}$
7.	$f(x,y) = ln\sqrt{x^2 + y^2}$	8.	$f(x,y) = 3y^3e^{-5x}$
For each function, find the partials. $a.\ \frac{\partial z}{\partial u}$ $b.\ \frac{\partial z}{\partial w}$			
9.	$z = (uw - 1)^4$	10.	$z = e^{\frac{(w^2 - u^2)}{2}}$

For each function, evaluate the stated partials.	
11.	$f(x,y) = 5x^3 + 4x^2y^2 - 3y^2$ $f_x(-1,2),\quad f_y(-1,2)$
12	$g(x,y) = e^{x^2 + y^2}$ $g_x(0,1),\quad g_y(0,1)$
13	$f(x,y) = \ln(x-y) + x^3y^2$ $f_x(2,1),\quad f_y(2,1)$
For each function of three variables, find the partials. $a.\ f_x \quad b.\ f_y \quad c.\ f_z$	
14.	$f(x,y,z) = xy^3z^2$
15.	$f(x,y,z) = 2e^{x^3 + y^3 + z^3}$
16.	$f(x,y,z) = (x^{0.5} + y^{0.5} + z^{0.5})^6$

	For each function, evaluate the stated partials.
17.	$h(x, y, z) = 4x^2z - 2xyz^2, \qquad h_x(3, -1, 2)$
18.	$g(x, y, z) = e^{4x^2 + 3y^2 + 2z^2}, \qquad g_z(1, 1, -1)$
19.	A home improvement company's profit ($P(x, y)$) function, from selling two products: (x) - Refrigerators and (y) - Stoves per day is given below. The number of products is given by: ($(x, y) = (150, 300)$) a. Find the marginal profit function for the Refrigerators? b. Find the marginal profit function for the Stoves? $P(x, y) = 3x^2 - 2xy + 4y^2 + 85x + 75y + 150$
20.	A company's production, is given by the "Cobb-Douglas" production function ($P(L, K)$), where (L) is the number of Labor Units, and (K) is the number of units of Capital. a. Find $P_L(30, 150)$, and interpret the answer? b. Find $P_K(30, 150)$, and interpret the answer? c. From your answers in (a) and (b), which will increase production more: an additional unit of Labor, or an additional unit of Capital? $P(L, K) = 160L^{\frac{1}{4}}K^{\frac{3}{4}}$

21.	Research finds that a person's upward mobility in life, depends on a person's education after college, and their income, based on the function ($M(x, y)$). Where (x) is the income in thousands of dollars, and (y) is years in education after college. a. Find $M_x(35, 8)$, and interpret the answer? b. Find $M_y(35, 8)$, and interpret the answer? $M(x, y) = 30x^{\frac{1}{5}}y^{\frac{1}{3}}$

Solutions

12. $a.\ 4x^3 + 10xy^2 + 1$ $b.\ 10x^2y - 8y - 1$

13. $a.\ 6(2x + 2y)^2$ $b.\ 6(2x + 2y)^2$

14. $a.\ 22x^{-0.9}y^{0.2}$ $b.\ 44x^{0.1}y^{-0.8}$

15. $a.\ 5x^{\frac{-3}{4}}y^{\frac{1}{5}}$ $b.\ 4x^{\frac{1}{4}}y^{\frac{-4}{5}}$

16. $a.\ \frac{4x^3}{x^4 + y^4}$ $b.\ \frac{4y^3}{x^4 + y^4}$

17. $a.\ 2ye^{2xy}$ $b.\ 2xe^{2xy}$

18. $a.\ \frac{x}{x^2 + y^2}$ $b.\ \frac{y}{x^2 + y^2}$

19. $a.\ -15y^3e^{-5x}$ $b.\ 9y^2e^{-5x}$

20. $a.\ 4w(uw - 1)^3$ $b.\ 4u(uw - 1)^3$

21. $a.\ -ue^{\frac{w^2 - u^2}{2}}$ $b.\ we^{\frac{w^2 - u^2}{2}}$

22. $a.\ f_x(-1, 2) = 15(-1)^2 + 8(-1)(2)^2 = -17$
$b.\ f_y(-1, 2) = 8(-1)^2(2) - 6(2) = 4$

23. $a.\ f_x(0, 1) = 2(0)e^{0^2 + 1^2} = 0$
$b.\ f_y(0, 1) = 2(1)e^{0^2 + 1^2} = 2e$

24. $a.\ f_x(2,1) = \frac{1}{(2-1)} + 3(2)^2(1)^2 = 13$

$b.\ f_y(2,1) = \frac{-1}{(2-1)} + 2(2)^3(1) = 15$

25. $a.\ y^3z^2$ $b.\ 3xy^2z^2$ $c.\ 2xy^3z$

26. $a.\ 6x^2e^{(x^3+y^3+z^3)}$ $b.\ 6y^2e^{(x^3+y^3+z^3)}$ $c.\ 6z^2e^{(x^3+y^3+z^3)}$

27. $a.\ \frac{3(x^{0.5}+y^{0.5}+z^{0.5})^5}{\sqrt{x}}$ $b.\ \frac{3(x^{0.5}+y^{0.5}+z^{0.5})^5}{\sqrt{y}}$

$c.\ \frac{3(x^{0.5}+y^{0.5}+z^{0.5})^5}{\sqrt{z}}$

28. $h_x(3,-1,2) = 8(3)(2) - 2(-1)(2)^2 = 56$

29. $g_z(1,1,-1) = 4(-1)e^{4(1)^2 + 3(1)^2 + 2(-1)^2} = -32{,}412.3$

30.

a. $P_x = 6x - 2y + 85$

b. $\$385$ *profit per additional Refrigerator*

c. $P_y = -2x + 8y + 75$

d. $\$2175$ *profit per additional Stove*

31.

a. $P_L(30, 150) = 40(30)^{\frac{-3}{4}}(150)^{\frac{3}{4}} = 134$ (The marginal productivity of labor is 134, so production increases by about 134 for each additional unit of labor)

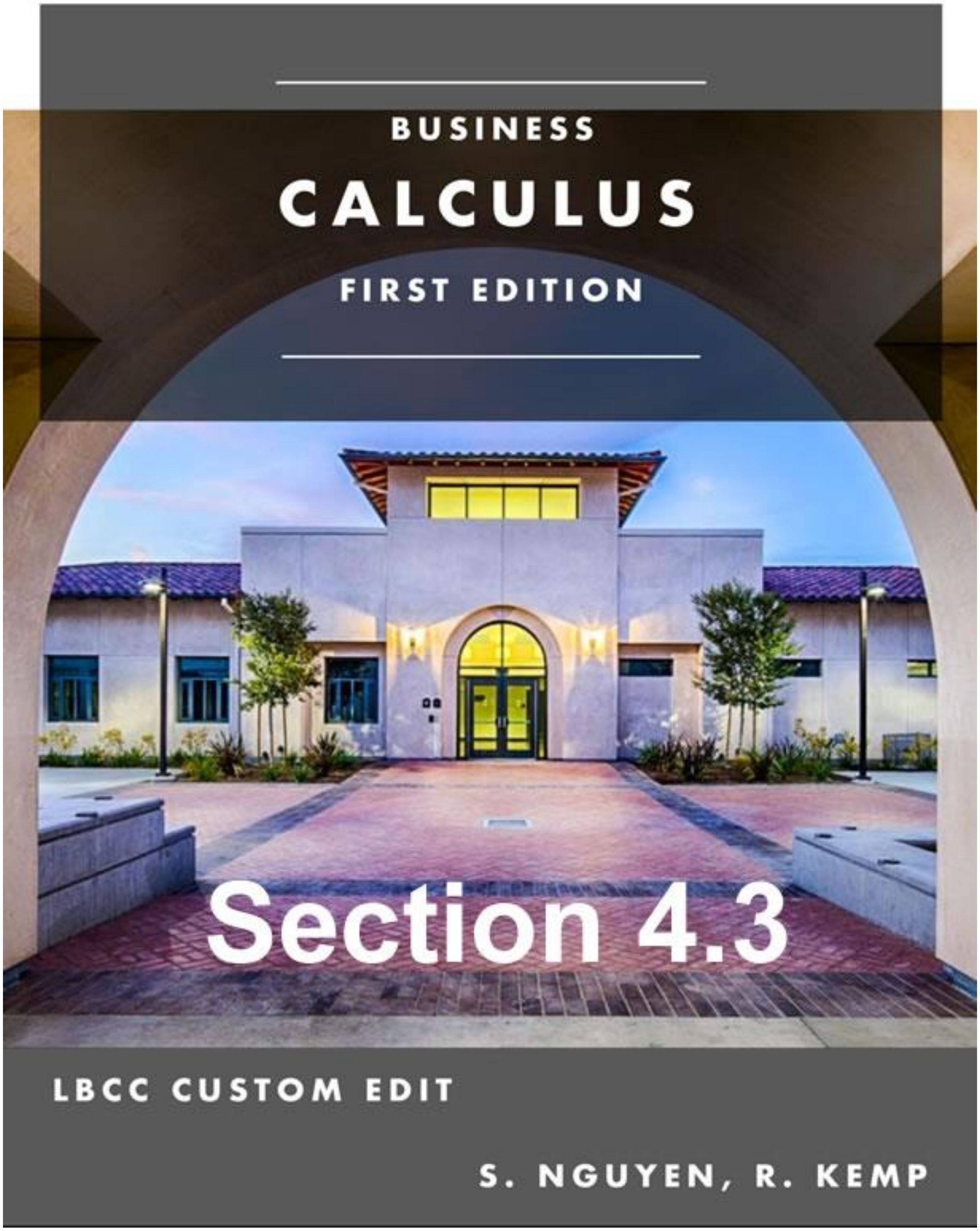
b. $P_K(30, 150) = 120(30)^{\frac{1}{4}}(150)^{\frac{-1}{4}} = 80$ (The marginal productivity of capital is 80, so production increases by about 80 for each additional unit of capital)

c. $Labor$

32.

a. $M_x(35, 8) = 6 \cdot (35)^{\frac{-4}{5}}(8)^{\frac{1}{3}} = 0.69$ (status increases by about 0.69 unit for each additional $1000 of income)

b. $M_y(35, 8) = 10 \cdot (35)^{\frac{1}{5}}(8)^{\frac{-2}{3}} = 5.09$ (status increases by about 5.09 unit for each additional year of education)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4.3 - HIGHER ORDER PARTIAL DERIVATIVES

Introduction

The partial derivatives tell us something about where a surface has local maxima and minima. Remember that even in the one-variable cases, there were critical points which were neither maxima nor minima. This is also true for functions of many variables.

"Higher Order" Partial Derivatives - Second Derivatives

When you find a partial derivative of a function of two variables, you get another function of two variables – you can take its partial derivatives, too. We've done this before, in the one-variable setting.

In the one-variable setting, the second derivative gave information about how the graph was curved.

In the two-variable setting, the second partial derivatives give some information about how the surface is curved, as you travel on cross-sections – but that's not very complete information about the entire surface.

Imagine that you have a surface that's ruffled around a point, like what happens near a button on an overstuffed sofa, or a pinched piece of fabric, or the wrinkly skin near your thumb when you make a fist.

Right at that point, every direction you move, something different will happen – it might increase, decrease, curve up, curve down…

A simple phrase like concave up or concave down can't describe all the things that can happen on a surface.

Surprisingly enough, though, there is still a second derivative test that can help you decide if a point is a local max or min or neither, so we still do want to find second derivatives.

Thus, we can differentiate a function more than once to obtain "Higher Order" derivatives.

Calculating a "Second Derivative" is a two-step process: For a given function (f), take the "First Derivative" of the function $(f' = \frac{\partial}{\partial x})$, then take the derivative of that same function again, that is the "Second Derivative" $(f'' = \frac{\partial^2}{\partial x^2})$.

First Order Partial Derivatives

The partial derivative of a function $(z = f(x, y))$ with respect to (x):

$$f_x = f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y)$$

The partial derivative of a function $(z = f(x, y))$ with respect to (y):

$$f_y = f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y)$$

Second Order Partial Derivatives

The partial derivative of a function $(f_x = f_x(x, y))$ with respect to (x):

$$f_{xx} = f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial^2}{\partial x^2} f(x, y)$$

The partial derivative of a function $(f_y = f_y(x, y))$ with respect to (y):

$$f_{yy} = f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial^2}{\partial y^2} f(x, y)$$

The partial derivative of a function $(f_x = f_x(x, y))$ with respect to (y):

$$f_{xy} = f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y)$$

The partial derivative of a function $(f_y = f_y(x, y)$ with respect to (x):

$$f_{yx} = f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y)$$

The functions (f_{yx}) and (f_{xy}) are called the "**mixed (second) partial derivatives**" of the function ($f(x, y)$).

Notice that the order of the variables for the "mixed partials" goes from right to left in the Leibniz notation instead of left to right.

Mixed Partial Derivative Theorem

If, the functions:

$$f(x, y) \quad , \quad f_x \quad , \quad f_y \quad , \quad f_{xy} \quad , \quad f_{yx}$$

are all continuous (no breaks in their graphs), then

$$f_{xy} = f_{yx}$$

$$f_{xy} = \frac{\partial^2}{\partial y \partial x} f(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y) = f_{yx}$$

$$f_{xy} = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial x} f_y(x, y) = f_{yx}$$

In fact, if (f) and all its appropriate partial derivatives are continuous, the "**mixed partials**" are equal even if they are of higher order, and even if the function has more than two variables.

This theorem means that the confusing Leibniz notation for second derivatives is not a big problem – in almost every situation the mixed partials are equal, so the order in which we compute them doesn't matter.

Example Problem #1

Find all Four Second Order Partial Derivatives

$$f(x,y) = y^4 + 2x^4y^5 - 3x^3y - x^4$$

Solution:

1. Calculate "**first order**" partial derivative of ($f(x,y)$) with respect to (x).

$$f_x = \frac{\partial}{\partial x}(y^4 + 2x^4y^5 - 3x^3y - x^4)$$

$$f_x = \frac{\partial}{\partial x}(y^4) + 2y^5\frac{\partial}{\partial x}(x^4) - 3y\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial x}(x^4)$$

$$f_x = 8x^3y^5 - 9x^2y - 4x^3$$

2. Calculate "**second order**" partial derivative of (f_x) with respect to (x).

$$f_{xx} = \frac{\partial}{\partial x}f_x = \frac{\partial}{\partial x}(8x^3y^5 - 9x^2y - 4x^3)$$

$$f_{xx} = 8y^5\frac{\partial}{\partial x}(x^3) - 9y\frac{\partial}{\partial x}(x^2) - 4\frac{\partial}{\partial x}(x^3)$$

$$f_{xx} = \frac{\partial^2}{\partial x^2}f(x,y) = 24x^2y^5 - 18xy - 12x^2$$

3. Calculate "**mixed second order**" partial derivative of (f_x) with respect to (x).

$$f_{xy} = \frac{\partial}{\partial y}f_x = \frac{\partial}{\partial y}(8x^3y^5 - 9x^2y - 4x^3)$$

$$f_{xy} = 8x^3\frac{\partial}{\partial y}(y^5) - 9x^2\frac{\partial}{\partial y}(y) - 4x^3\frac{\partial}{\partial y}(1)$$

$$f_{xy} = \frac{\partial^2}{\partial y\partial x}f(x,y) = 40x^3y^4 - 9x^2$$

Example Problem #1 – Cont'd

Find all Four Second Order Partial Derivatives

$$f(x,y) = y^4 + 2x^4y^5 - 3x^3y - x^4$$

Solution:

4. Calculate the "**first order**" partial derivative of ($f(x,y)$) with respect to (y).

$$f_y = \frac{\partial}{\partial y}(y^4 + 2x^4y^5 - 3x^3y - x^4)$$

$$f_y = \frac{\partial}{\partial y}(y^4) + 2x^4\frac{\partial}{\partial y}(y^5) - 3x^3\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(x^4)$$

$$f_y = 4y^3 + 10x^4y^4 - 3x^3$$

5. Calculate "**second order**" partial derivative of (f_y) with respect to (y).

$$f_{yy} = \frac{\partial}{\partial y}f_y = \frac{\partial}{\partial y}(4y^3 + 10x^4y^4 - 3x^3)$$

$$f_{yy} = \frac{\partial}{\partial y}(4y^3) + 10x^4\frac{\partial}{\partial y}(y^4) - 3\frac{\partial}{\partial y}(x^3)$$

$$f_{yy} = \frac{\partial^2}{\partial y^2}f(x,y) = 12y^2 + 40x^4y^3$$

6. Calculate "**mixed second order**" partial derivative of (f_y) with respect to (x).

$$f_{yx} = \frac{\partial}{\partial x}f_y = \frac{\partial}{\partial x}(4y^3 + 10x^4y^4 - 3x^3)$$

$$f_{yx} = 4\frac{\partial}{\partial x}(y^3) + 10y^4\frac{\partial}{\partial x}(x^4) - 3\frac{\partial}{\partial x}(x^3)$$

Notice that the "**mixed partials**" are equal:

$$f_{xy} = f_{yx} = 40x^3y^4 - 9x^2$$

Example Problem #2

Find all Four Second Order Partial Derivatives

$$f(x,y) = x^2 - 4xy + 4y^2$$

Solution:

1. Calculate "**first order**" partial derivatives of $(f(x,y))$ with respect to (x).

$$f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(x^2 - 4xy + 4y^2) = 2x - 4y$$

$$f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(x^2 - 4xy + 4y^2) = -4x + 8y$$

2. Calculate "**second order**" partial derivatives.

$$f_{xx} = \frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(2x - 4y) = 2$$

$$f_{yy} = \frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(-4x + 8y) = 8$$

3. Calculate "**mixed second order**" partial derivatives.

$$f_{xy} = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(2x - 4y) = -4$$

$$f_{yx} = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(-4x + 8y) = -4$$

4.3 - EXERCISES

	For each function, find the second-order partials. $a.\ f_{xx}$ $b.\ f_{xy}$ $c.\ f_{yy}$ $d.\ f_{yx}$
1.	$f(x,y) = 4y^3 + 5x^3y^2 - 3x^4$
2.	$f(x,y) = 12x^{\frac{1}{4}}y^{\frac{1}{2}} - 8x^2y^3$
3.	$f(x,y) = y\ ln(x) - xe^y$
4.	$f(x,y) = 5x^2 + 6y^2 - 10xy + 4x - 7y + 9$
5.	$f(x,y) = \sqrt{x+y}$

For each function, find the partial derivatives. $a.\ f_{xxx}$ $b.\ f_{yyy}$ $c.\ f_{xyx}$ $d.\ f_{xxy}$	
6.	$f(x,y) = e^{2x} - x^3y^5$
7.	$f(x,y) = \ln(xy) + 4x^3 - 9y^5$

Solutions

33.

a. $30xy^2 - 36x^2$

b. $30x^2y$

c. $24y + 10x^3$

d. $30x^2y$

34.

a. $-\frac{9}{4}x^{\frac{-7}{4}}y^{\frac{1}{2}} - 16y^3$

b. $\frac{3}{2}x^{\frac{-3}{4}}y^{\frac{-1}{2}} - 48xy^2$

c. $-3x^{\frac{1}{4}}y^{\frac{-3}{2}} - 48x^2y$

d. $\frac{3}{2}x^{\frac{-3}{4}}y^{\frac{-1}{2}} - 48xy^2$

35.

a. $-\frac{y}{x^2}$

b. $\frac{1}{x} - e^y$

c. $-xe^y$

d. $\frac{1}{x} - e^y$

36.

a. 10

b. -10

c. 12

d. -10

37.

a. $\frac{-1}{4(x+y)^{\frac{3}{2}}}$

b. $\frac{-1}{4(x+y)^{\frac{3}{2}}}$

c. $\frac{-1}{4(x+y)^{\frac{3}{2}}}$

d. $\frac{-1}{4(x+y)^{\frac{3}{2}}}$

6.

a. $8e^{2x} - 6y^5$

b. $-60x^3y^2$

c. $-30xy^4$

d. $-30xy^4$

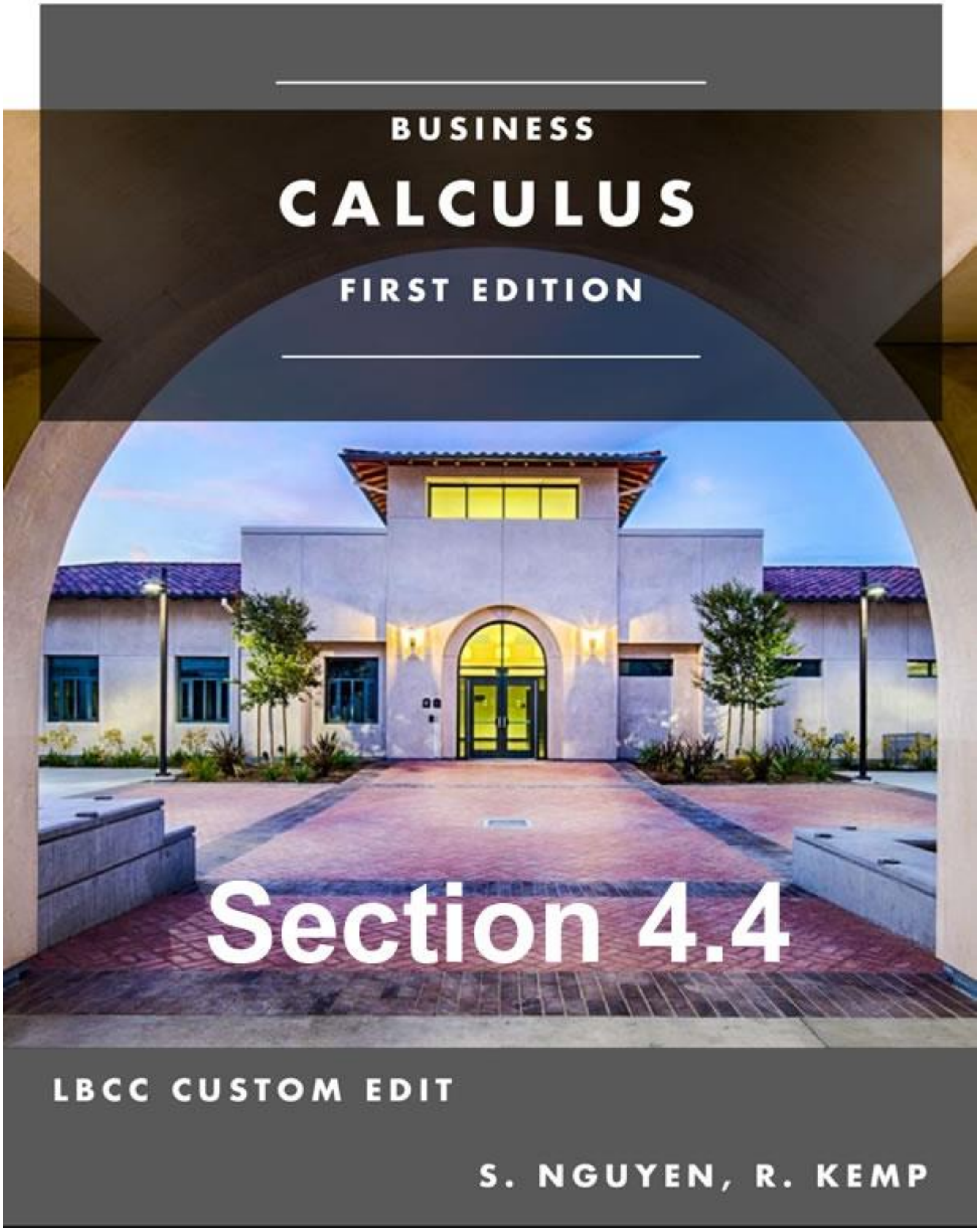
7.

a. $\frac{2}{x^3} + 24$

b. $\frac{2}{y^3} - 540y^2$

c. 0

d. 0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4.4 - OPTIMIZING FUNCTIONS OF MULTI-VARIABLES

Introduction

The partial derivatives tell us something about where a surface has local maxima and minima. In this section, we will see how to maximize and minimize functions by finding the critical points and then performing second derivative tests as a way of optimizing. Such optimizing calculations have many applications in business such as maximizing profit and revenue, or minimizing the cost for a company that produces and sells more than one product, hence multiple variables.

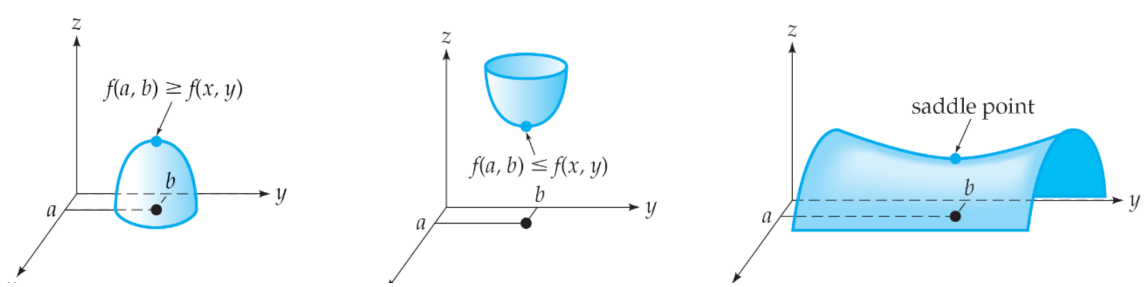
Local Maximum, Local Minimum and Saddle Points

Let's briefly review max-min problems in one variable.

A local max is a point on a curve that is higher than all the nearby points.

A local min is lower than all the nearby points.

We know that local max or min can only occur at critical points, where the derivative is zero or undefined.



- "Left graph"; The function $(f(x, y))$ has a local *maximum* value at (*a, b*).
- "Center graph"; The function $(f(x, y))$ has a local *minimum* value at (*a, b*)
- "Right graph"; The function $(f(x, y))$ has a saddle point at (a, b) - (which is neither a maximum nor a minimum value)

Critical Points

The “Critical Points” for a single variable function ($f(x)$) are the points where the dericative is equal to 0 or is undefined. They lead to possible local mins or maxs.

For functions with two variables ($f(x,y)$) the “Critical Points” lead to possible Local “Relative” Maximums, Local “Relative” Minimums, and Saddle Points.

Definition for Local (Relative) Maximum & Minimum Points

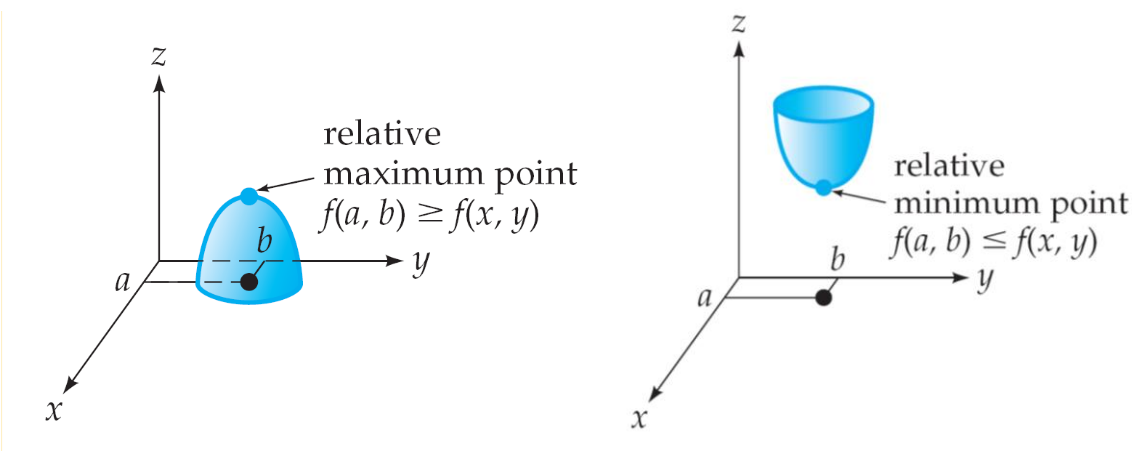
Local (Relative) Maximum & Minimum Point - Definition:

The function (f) has a **Local “Relative” Maximum** point at (a, b), if;

$$f(a,b) \geq f(x,y) \quad ; \quad \text{for all points } (x,y) \text{ near (a, b)}$$

The function (f) has a **Local “Relative” Minimum** point at (a, b), if

$$f(a,b) \leq f(x,y) \quad ; \quad \text{for all points } (x,y) \text{ near (a, b)}$$



A point on a surface is a “**Local “Relative” Maximum**” if it’s higher than all the points nearby.

A point on a surface is a “**Local “Relative” Minimum**” if it’s lower than all the points nearby.

The “Critical Points” of a function ($f(x)$) are found by setting the “first derivative” of the function $f' = 0 \; or \; Undefined$. The situation with a function of two variables is much the same.

Just as in the one-variable case, the first step is to find the “Critical Points”, places where both the partial derivatives are: either zero or undefined.

The “Partial Derivative” definition for “Critical Points” on a surface is:

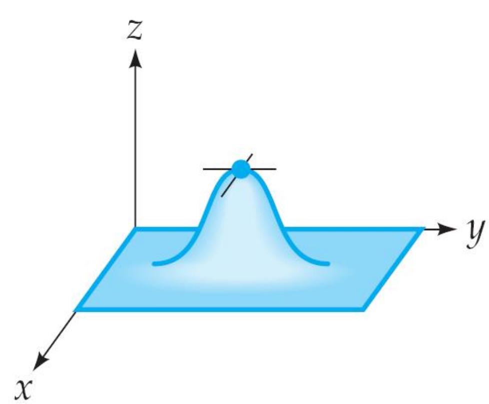
Two Variable - Partial Derivative - Critical Point - Definition:

A “**Critical Point**” (x, y), of a function ($f(x, y)$) is a point (x, y), where both the following are true:

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = 0 \;\; or \;\; Undefined$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = 0 \;\; or \;\; Undefined$$

Just as in the one-variable case, a local max or min of (f) can only occur at a critical point.

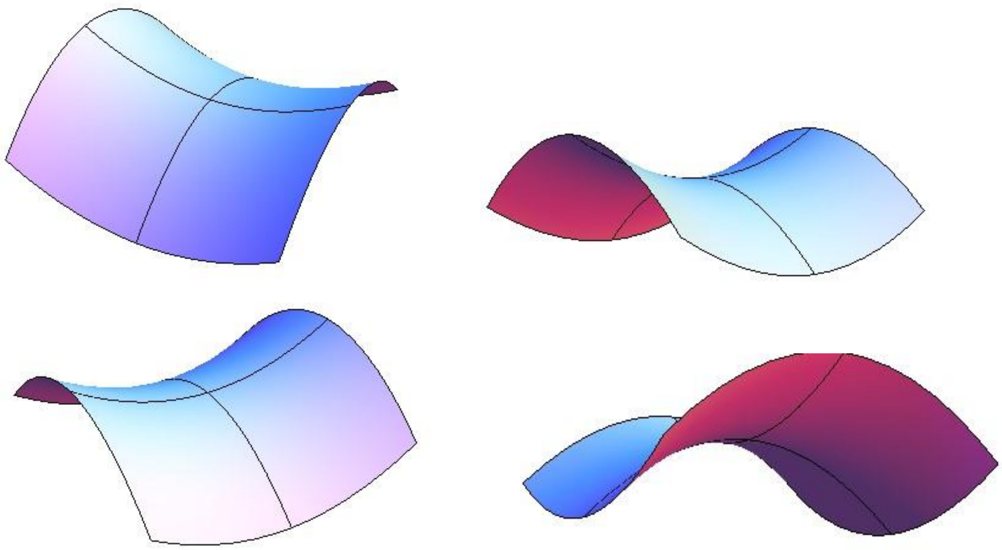


As you can see from the image, at a local maximum the derivatives in both the x- and y-direction must be equal to 0, i.e the tangent line must be horizontal in both x- and y-directions. The same would be true for a local minimum, or a saddle point which we we will be talking about more in detail on the next page.

A **saddle point** is a point on a surface that is a minimum along some paths, and a maximum along some others. It's called this because it's shaped a bit like a saddle you might use to ride a horse.

You can see a saddle point by making a fist – between the knuckles of your index and middle fingers, you can see a place that is a minimum as you go across your knuckles, but a maximum as you go along your hand toward your fingers.

Here is a picture of a saddle point from a few different angles.



If we think of a surface ($z = f(x, y)$) as a landscape, then relative maximum and minimum points correspond to "hilltops" and "valley bottoms," and a saddle point corresponds to a "mountain pass" between two peaks.

The lines show what the surface looks like above the (x-axis) and (y-axis).

Notice how the point above the origin, where the lines cross, is a local minimum in one direction, but a local maximum in the other direction. Also notice the the tangent lines to the saddle point would be horizontal lines, hence the slopes of these tangent lines would equal to 0. So, a Saddle Point would satisfy the conditions of a Critical Point that both partial derivatives would equal to 0.

Second Derivative Test for Functions with Two Variables

Just as in the one-variable case, we will need a way to test critical points to see whether they are a local max or min, or possibly a saddle point.

There is a second derivative test for functions of two variables that can help, but, just as in the one-variable case, it won't always give an answer.

The equivalent of the second derivative test for one variable functions, which we discussed in section 2.2 is called the **D-Test** for functions with two variables.

Just as in the case of the second derivative test for one variable functions, we will still need to find the critical numbers first, as any local min or max can only occur at a critical point.

We will then have to find the second derivative, which in the case of a function in two variables consists ot 4 partial derivatives . So, we will need some type of a rule that involves all these partial derivatives, rather than just the one second derivative $f''(x)$ that we had before. The D - function described below acomplishes just that.

The Second Derivative Test for Functions of Two Variables

1. Find the critical points for function – ($f(x, y)$)
2. Find all second order partial derivatives.: $f_{xx},\ f_{yy},\ f_{xy} = f_{yx}$
3. Set up the D-Test Function:

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - \left[f_{xy}(x, y)\right]^2$$

4. Evaluate the D-Test function at each of the critical points, say (a, b)

$$D(a, b) = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$$

We will draw our conclusion based on the sign of the value we get from this test.

- If ($D(a, b) > 0$)
  - then ($f(x, y)$) has a **local max or min**, at the critical point.

  To see which, look at the sign of (f_{xx}).

  - If ($f_{xx}(a, b) > 0$) and ($D(a, b) > 0$)
    - then ($f(x, y)$) has a "**local minimum**" at the critical point.
  - If ($f_{xx}(a, b) < 0$) and ($D(a, b) > 0$
    - then ($f(x, y)$) has a "**local maximum**" at the critical point.
- If ($D(a, b) < 0$)
  - then ($f(x, y)$) has a "**Saddle Point**", at the critical point (Neither, No Maximum, and No Minimum value).
- If ($D(a, b) = 0$)
  - then ($f(x, y)$) is **inconclusive**, there could be a local max, local min, or neither (i.e., the test in inconclusive).

Example Problem #1

Find Critical Points

Find the Critical Points for the function:

$$z = f(x,y) = 3xy - 2x^2 - 2y^2 + 14x - 7y - 5$$

Solution:

1. Find First Order Partial Derivative:

$$f_x = -4x + 3y + 14 \quad ; \quad f_y = 3x - 4y - 7$$

2. Set Partials equal to zero:

$$f_x = -4x + 3y + 14 = 0$$

$$f_y = 3x - 4y - 7 = 0$$

3. Next, solve the above equations by the technique of solving systems of linear equations. equations. We multiply both sides of (f_x) by (3). And multiply both sides of (f_y) by (4); so, that the (x) term drops out when we add.

$$-12x + 9y + 42 = 0$$

$$12x - 16y - 28 = 0$$

$$-7y + 14 = 0$$

$$y = 2$$

4. Next, substitute ($y = 2$) into either equation above to get the value for (x).

$$-12x + 9(2) + 42 = 0$$

$$12x = 60$$

$$x = 5$$

5. The Critical Points ($CP: (x,y)$) are given:

$$CP: (x,y) = (5,2)$$

Example Problem #2

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

Using the results from the Example #1, above;

Find all relative extreme values for the function:

$$z = f(x,y) = 3xy - 2x^2 - 2y^2 + 14x - 7y - 5$$

Solution:

4. Calculate "**first order**" partial derivatives of ($f(x,y)$) with respect to (x).

$$f_x = \frac{\partial}{\partial x} f(x,y) = -4x + 3y + 14$$

$$f_y = \frac{\partial}{\partial y} f(x,y) = 3x - 4y - 7$$

5. Calculate "**second order**" partial derivatives.

$$f_{xx} = \frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(-4x + 3y + 14) = -4$$

$$f_{yy} = \frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(3x - 4y - 7) = -4$$

6. Calculate "**mixed second order**" partial derivatives.

$$f_{xy} = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(-4x + 3y + 14) = 3$$

$$f_{yx} = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(3x - 4y - 7) = 3$$

Example Problem #2 – Cont'd

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

7. Second Order Partials.

$$f_{xx} = -4$$

$$f_{yy} = -4$$

$$f_{xy} = 3$$

D-Calculation:

$$D(x,y) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$$

$$D(x,y) = (-4)(-4) - [3]^2 = 16 - 9 = 7$$

$$D(x,y) = D(5,2) = 7$$

D-Test:

$$D(5,2) = 7 > 0 \quad ; \quad f_{xx}(5,2) = -4 < 0$$

Since ($D > 0$) is positive, and ($f_{xx} < 0$) is negative, ($f(x,y)$) has a "**Relative Maximum**" value at the "Critical Point" (CP: $(5,2)$)

8. Find the "**Relative Maximum**" value, by evaluating ($f(x,y)$) at the critical point (CP: $(5,2)$).

$$z = f(x,y) = 3xy - 2x^2 - 2y^2 + 14x - 7y - 5$$

$$z = f(5,2) = 3(5)(2) - 2(5)^2 - 2(2)^2 + 14(5) - 7(2) - 5$$

$$z = f(5,2) = 30 - 50 - 8 + 70 - 14 - 5$$

$$z = f(5,2) = 23$$

The "**Relative Maximum**" value is $z = 23$ at $x = 5,$ $y = 2$

The "**Relative Maximum**" value: $(x\ ,\ y\ , f(x,y)) = (5, 2, 23)$

Example Problem #3

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

Find all Relative Extreme Values:

$$f(x,y) = x^2 + y^3 - 8x - 27y$$

Solution:

1. Find First Order Partial Derivative:

$$f_x = 2x - 8 \quad ; \qquad f_y = 3y^2 - 27$$

2. Set Partials equal to zero:

$$f_x = 2x - 8 = 0$$

$$f_y = 3y^2 - 27 = 0$$

3. Next, solve the above partials.

The first yields a x-value

$$2x = 8$$

$$x = 4$$

The second yields a y-value

$$3y^2 - 27 = 0$$

$$y^2 = 9$$

$$y = \pm 3$$

4. Since ($y = \pm 3$) is two values, there are two “Critical Points” ($CP: (x,y)$)

$$CP_1: (4,3) \quad \text{and} \quad CP_2: (4,-3)$$

Example Problem #3 – Cont'd

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

Find all Relative Extreme Values:

$$f(x,y) = x^2 + y^3 - 8x - 27y$$

5. Calculate "**second order**" partial derivatives.

$$f_{xx} = \frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(2x - 8) = 2$$

$$f_{yy} = \frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(3y^2 - 27) = 6y$$

Since ($y = \pm 3$)

$$f_{yy}(4,3) = 18 \quad ; \quad f_{yy}(4,-3) = -18$$

6. Calculate "**mixed second order**" partial derivatives.

$$f_{xy} = f_{yx} = 0$$

7. Second Order Partials.

$$f_{xx} = 2$$

$$f_{yy}(4,3) = 18 \quad ; \quad f_{yy}(4,-3) = -18$$

$$f_{xy} = 0$$

Example Problem #3 – Cont'd

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

8. We apply the D-Test to the Critical Points one at a time.

D-Calculation:

$$D(x, y) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$$

At Critical Point: $CP_1: (x, y) = CP: (4, 3)$

$$D(4, 3) = (2)(18) - [0]^2$$

$$D(4, 3) = 36 > 0$$

$$f_{xx}(4, 3) = 2 > 0$$

Since $(D > 0)$ is positive, and $(f_{xx} > 0)$ is positive, so $(f(x, y))$ has a "**Relative Minimum**" value at the "Critical Point" $(CP: (x, y) = CP_1: (4, 3))$

At Critical Point: $CP_2: (x, y) = CP: (4, -3)$

$$D(4, -3) = (2)(-18) - [0]^2$$

$$D(4, -3) = -36 < 0$$

Since $(D < 0)$ is negative, therefore "**Saddle Point**", so $(f(x, y))$ has a "**No Relative Extreme**" value at the "Critical Point" $(CP: (x, y) = CP_2: (4, -3))$.

Example Problem #3 – Cont'd

Find All Relative Extreme Values (Local Maximum, Local Minimum, and Saddle Points) of a Function

9. Find the "**Relative Minimum**" value, by evaluating ($f(x, y)$) at the critical value (CP: $(4, 3)$).

$$z = f(x, y) = x^2 + y^3 - 8x - 27y$$

$$z = f(4, 3) = (4)^2 + (3)^3 - 8(4) - 27(3)$$

$$z = f(4, 3) = 16 + 27 - 32 - 81$$

$$z = f(4, 3) = -70$$

The "**Relative Minimum**" value: $z = -70$ at $x = 4$, $y = 3$

The "**Saddle Point**" value: $x = 4$, $y = -3$

Or

The "**Relative Minimum**" value: $(x, y, f(x, y)) = (4, 3, -70)$

The "**Saddle Point**" value: $(x, y, f(x, y)) = (4, -3, Undefined)$

Maximizing Profit

Similarly to the problems we solved in Chapter 2 regarding Profit, Revenue and Cost, we will solve the same type of problems now, the only difference being that in our problems now we will have two types of items being produced, rather than just one.

In one such example in Chapter we explored the idea that the price at which we sell one item would depend on the number of itesm sold.

Suppose that a company produces two products, (A) and (B):

The two "**Price**" functions are:

$$p(x) \;=\; \begin{pmatrix} \textit{Price at which exactly } (x) \textit{ units} \\ \textit{of product } (A) \textit{ will be sold} \end{pmatrix}$$

$$q(y) \;=\; \begin{pmatrix} \textit{Price at which exactly } (y) \textit{ units} \\ \textit{of product } (B) \textit{ will be sold} \end{pmatrix}$$

The two "**Revenue**" functions are:

$$R(x) \;=\; p(x)\cdot x \;=\; \begin{pmatrix} \textit{Revenue at which exactly } (x) \textit{ units} \\ \textit{of product } (A) \textit{ will generate} \end{pmatrix}$$

$$R(y) \;=\; q(y)\cdot y \;=\; \begin{pmatrix} \textit{Revenue at which exactly } (y) \textit{ units} \\ \textit{of product } (B) \textit{ will generate} \end{pmatrix}$$

If the Company's "**Cost**" function is given - $C(x, y)$.

The Company's "**Profit**" function is given by $P(x, y)$

$$P(x, y) \;=\; R(x) \;+\; R(y) \;-\; C(x, y)$$

$$P(x, y) \;=\; p(x)\cdot x \;+\; q(y)\cdot y \;-\; C(x, y)$$

Example Problem #4

Maximizing Profit for a Company

A company manufactures two products: An **Electric Skateboard** and an **Electric Bicycle**.

The price function for the **Electric Skateboard** is: $p(x) = 18 - 2x$

The price function for the **Electric Bicycle** is: $q(y) = 27 - y$

Both price functions ($p(x)$) and ($q(y)$) are in hundred dollars, where (x) and (y) are the numbers Skateboard and Bicycles produced per hour.

The Company's Cost function ($C(x,y)$) is given to be:

$$C(x,y) = 15x + 12y - xy + 6$$

Find the quantities and the prices of the two products that maximize profit and find the maximum profit.

Solution:

1. Find Revenue functions:

$$R(x) = p(x) \cdot x = 18x - 2x^2$$

$$R(y) = q(y) \cdot y = 27y - y^2$$

2. Calculate Profit Function:

$$P(x,y) = R(x) + R(y) - C(x,y)$$

$$P(x,y) = p(x) \cdot x + q(y) \cdot y - C(x,y)$$

$$P(x,y) = 18x - 2x^2 + 27y - y^2 - 15x - 12y + xy - 6$$

$$P(x,y) = -2x^2 - y^2 + 15y + 3x + xy - 6$$

Example Problem #4 – Cont'd

Maximizing Profit for a Company

3. Find the Partials of the Profit function:

$$P_x = \frac{\partial}{\partial x}P(x,y) = \frac{\partial}{\partial x}(-2x^2 - y^2 + 15y + 3x + xy - 6)$$

$$P_x = -4x + y + 3$$

$$P_y = \frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial y}(-2x^2 - y^2 + 15y + 3x + xy - 6)$$

$$P_y = x - 2y + 15$$

4. Next, we multiply both sides of (P_x) by (2); so, that the (y) term drops out when we add.

$$-8x + 2y + 6 = 0$$

$$x - 2y + 15 = 0$$

$$\overline{\qquad\qquad\qquad\qquad\qquad}$$

$$-7x + 21 = 0$$

$$x = 3$$

5. Substituting ($x = 3$) into either equation above give the value for (y).

$$-8(3) + 2y + 6 = 0$$

$$2y = 18$$

$$y = 9$$

6. The Critical Points ($CP: (x,y)$) are given:

$$CP: (x,y) = CP: (3,9)$$

Example Problem #4 – Cont'd

Maximizing Profit for a Company

7. Calculate "**second order**" partial derivatives.

$$P_{xx} = \frac{\partial^2}{\partial x^2}P(x,y) = \frac{\partial}{\partial x}P_x = \frac{\partial}{\partial x}(-4x + y + 3) = -4$$

$$P_{yy} = \frac{\partial^2}{\partial y^2}P(x,y) = \frac{\partial}{\partial y}P_y = \frac{\partial}{\partial y}(x - 2y + 15) = -2$$

8. Calculate "**mixed second order**" partial derivatives.

$$P_{xy} = P_{yx} = 1$$

$$\frac{\partial}{\partial y}(-4x + y + 3) = \frac{\partial}{\partial x}(x - 2y + 15) = 1$$

9. We apply the D-Test to the Critical Points one at a time.

D-Calculation:

$$D(x,y) = P_{xx} \cdot P_{yy} - \left[P_{xy}\right]^2$$

At Critical Point: $CP: (x,y) = CP: (3,9)$

$$D(3,9) = (-4)(-2) - [1]^2$$

$$D(3,9) = 7 > 0$$

$$P_{xx} = -4 < 0$$

Since $(D > 0)$ is positive, and $(P_{xx} < 0)$ is negative, so $(P(x,y))$ has a "**Relative Maximum**" value at the "Critical Point" ($CP: (x,y) = CP: (3,9)$)

Therefore, the "**Profit**" is **Maximized**!

Example Problem #4 – Cont'd

Maximizing Profit for a Company

10. The "**Profit**" is determined from the "**Profit**" function:

$$P(x,y) = -2x^2 - y^2 + 15y + 3x + xy - 6$$

$$P(3,9) = -2(3)^2 - (9)^2 + 15(9) + 3(3) + (3)(9) - 6$$

$$P(3,9) = -18 - 81 + 135 + 9 + 27 - 6 = 66$$

$$P(3,9) = 66 \ (Hundred\ Dollars) = \$6{,}600$$

11. The "**Prices**" for Products (A) and (B):

The price function for the **Electric Skateboard** is:

$$p(x) = 18 - 2x$$

$$p(3) = 18 - 2(3) = 12$$

$$p(3) = 12 \ (Hundred\ Dollars) = \$1{,}200$$

The price function for the **Electric Bicycle** is:

$$q(y) = 27 - y$$

$$q(9) = 27 - 9 = 18$$

$$q(9) = 18 \ (Hundred\ Dollars) = \$1{,}800$$

The Profit is Maximized when the company produces (3 Skateboards) per hour, selling them for ($1,200) each, and (9 Electric Bicycles) per hour selling them for ($1,800) each. The maxim Profit will be ($6,600) per hour.

Example Problem #5

Maximizing Revenue for a Company

A company manufactures two products. The **demand equations** for the two products are:

$$q_1 = 200 - 3p_1 - p_2$$

$$q_2 = 150 - p_1 - 2p_2$$

where $p_1 and\ p_2$ are prices for product 1 and product 2, respectively.

Find the price the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

Solution:

1. Find Revenue functions:

Revenue is still price × quantity. If we're selling two products, the total revenue will be the sum of the revenues from the two products.

$$R(p_1, p_2) = p_1 q_1 + p_2 q_2$$

$$R(p_1, p_2) = p_1(200 - 3p_1 - p_2) + p_2(150 - p_1 - 2p_2)$$

$$R(p_1, p_2) = 200p_1 - 3p_1^2 - 2p_1 p_2 + 150p_2 - 2p_2^2$$

2. Find the Partials of the Revenue function:

$$R_{p_1}(p_1, p_2) = \frac{\partial}{\partial p_1} R(p_1, p_2) = 200 - 6p_1 - 2p_2$$

$$R_{p_2}(p_1, p_2) = \frac{\partial}{\partial p_2} R(p_1, p_2) = 150 - 2p_1 - 4p_2$$

Example Problem #5 – Cont'd

Maximizing Revenue for a Company

3. Next, set the above partials equal to 0 to find the critical points.

$$200 - 6p_1 - 2p_2 = 0$$
$$150 - 2p_1 - 4p_2 = 0$$

$$6p_1 + 2p_2 = 200$$
$$2p_1 + 4p_2 = 150$$

4. Next, we multiply both sides of $(R_{p_2}(p_1, p_2))$ by (-2); so, that the (p_2) term drops out when we add.

$$-12p_1 - 4p_2 = -400$$
$$2p_1 + 4p_2 = 150$$

$$-10p_1 = -250$$
$$p_1 = \frac{-250}{-10} = 25$$

5. Next, substitute $(p_1 = 25)$ into one of the equations above.

$$2p_1 + 4p_2 = 150$$
$$2(25) + 4p_2 = 150$$
$$p_2 = \frac{150 - 50}{4} = 25$$

Therefore, the critical "Price" points are given below:

$$CP: \quad (p_1, p_2) = (\$25, \$25)$$

Example Problem #5 – Cont'd

Maximizing Revenue for a Company

6. Calculate "**second order**" partial derivatives.

$$R_{p_1p_1}(p_1,p_2) = \frac{\partial}{\partial p_1}(200 - 6p_1 - 2p_2) = -6$$

$$R_{p_2p_2}(p_1,p_2) = \frac{\partial}{\partial p_2}(150 - 2p_1 - 4p_2) = -4$$

Calculate "**mixed second order**" partial derivatives.

$$R_{p_1p_2}(p_1,p_2) = \frac{\partial}{\partial p_2}(200 - 6p_1 - 2p_2) = -2$$

$$R_{p_2p_1}(p_1,p_2) = \frac{\partial}{\partial p_1}(150 - 2p_1 - 4p_2) = -2$$

7. We apply the D-Test to the Critical Points one at a time.

D-Calculation:

$$D = (-6)(-4) - (-2)^2 = 20$$

At Critical Point: $(25, 25)$

$$D(25,25) = 20 > 0$$

$$R_{p_1p_1}(25,25) = -6 < 0$$

Since $(D > 0)$ is positive, and $R_{p_1p_1} < 0)$ is negative, the function has a "**Relative Maximum**" value at the "Critical Point" (CP: $(25, 25)$).

Example Problem #5 – Cont'd

Finding the Revenue or Consumer Expenditure

8. The "**Revenue**" is determined from the "**Revenue**" function:

$$R(p_1, p_2) = 200p_1 - 3p_1^2 - 2p_1p_2 + 150p_2 - 2p_2^2$$

$$R(25, 25) = 200(25) - 3(25)^2 - 2(25)(25) + 150(25) - 2(25)^2$$

$$R(25, 25) = 5000 - 1875 - 1250 + 3750 - 1250 = 4375$$

$$R(25, 25) = \$4{,}375$$

To maximize revenue, the company should charge $25 per unit for both products. This will yield a maximum revenue of $4375.

9. The "**Demand**" for Products (1) and (2):

The Demand function for the **Product (1)** is:

$$q_1 = 200 - 3p_1 - p_2$$

$$q_1 = 200 - 3(25) - 25 = 100$$

$$q_1 = 100\, units$$

The Demand function for the **Product (2)** is:

$$q_2 = 150 - p_1 - 2p_2$$

$$q_2 = 150 - 25 - 2(25) = 75$$

$$q_2 = 75\, units$$

4.4 - EXERCISES

Find the relative extreme values of each function using the D -Test.	
1.	$f(x, y) = 6y^2 + 4x^2 + 4xy - 16y + 8x$
2.	$f(x, y) = 4xy + 12x - 6y - 2x^2 - 3y^2 - 6$
3.	$f(x, y) = x^2 - 4y^2 + 4xy + 4x + 8y + 8$
4.	$f(x, y) = 6x - 10y + 2xy + 2$
5.	$f(x, y) = x^3 + 3xy + y^3$
6.	$f(x, y) = -y^2 + x^3 + 2y - 3x$
7.	$f(x, y) = 2x^2 - 2y^3 + 24y + 4x$
8.	$f(x, y) = 4y - x^3 + 2xy$
9.	$f(x, y) = 2\, lnx + lny - 4x - y$
10.	$f(x, y) = e^{x^2 + y^2 - 4x}$

11.	A company manufactures two types of products A and B. The price function for product (A) is ($p = 30 - x$) and for product (B) the price is ($q = 25 - y$) both prices are given in dollars, where (x) and (y) are the amounts of products (A) and (B) respectively. If the Cost function is given by: $$C(x, y) = 9y + 4x + xy + 20 \rightarrow dollars$$ d. Find the Revenue function from selling these items. e. Find the Profit function. f. Perform D-Test. g. Find the quantities and prices of the two products that maximize profit. h. Find the Maximum Profit value.
12	A clothing company sells two types of shoes slip-ons and high-tops. The price function for slip-ons is ($p = 50 - 2x$) dollars for ($0 \leq x \leq 25$). The price function for high-tops is ($q = 100 - 4y$) dollars for ($0 \leq y \leq 25$). Where (x) and (y) are the numbers slip-ons and high-tops sold per day. The company Cost of producing each slip-on is $10/shoe and the cost of producing each high-top is $12/shoe. Also the fixed costs are $100/day. a. Find the Revenue function from selling these two types of shoes. b. Find the Cost function. c. Find the Profit function. d. Perform D-Test. e. Find how many shoes and at the prices at which they should be sold to maximize the daily profit. f. Find the maximum daily profit.

13.	The demand functions for two products are given below. $p_1, p_2, q_1,\ and\ q_2$ are the prices (in dollars) and quantities for products 1 and 2. $$q_1 = 200 - 3p_1 + p_2$$ $$q_2 = 150 + p_1 - 2p_2$$ Find the price and the quantities the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

Solutions

38. $Rel. Min. value: \quad f(x,y) = -24 \quad at \quad x = -2, \; y = 2$

39. $Rel. Max. value: \quad f(x,y) = 21 \quad at \quad x = 6, \; y = 3$

40. $No\ Rel. Extreme\ values: \quad Saddle\ Point:$
$f(x,y) = 4 \quad at \quad x = -2, \; y = 0$

41. $No\ Rel. Extreme\ values: \quad Saddle\ Point$
$f(x,y) = 32 \quad at \quad x = 5, \; y = -3$

42. $Rel. Max. value: \quad f(x,y) = 1 \quad at \quad x = -1, \; y = -1$

$Saddle\ Point \quad at \quad x = 0, \; y = 0$

43. $Rel. Max. value: \quad f(x,y) = 3 \quad at \quad x = -1, \; y = 1$

$Saddle\ Point \quad at \quad x = 1, \; y = 1$

44. $Rel. Min. value: \quad f(x,y) = -34 \quad at \quad x = -1, \; y = -2$

$Saddle\ Point \quad at \quad x = -1, \; y = 2$

45. $No\ Rel. Extreme\ values: \quad Saddle\ Point$
$f(x,y) = 8 \quad at \quad x = -2, \; y = 6$

46. $Rel. Max. value: \quad f(x,y) = \ln\left(\frac{1}{4}\right) - 3 \quad at \quad x = \frac{1}{2}, \; y = 1$

47. $Rel. Min. value: \quad f(x,y) = e^{-4} \quad at \quad x = 2, \; y = 0$

48.

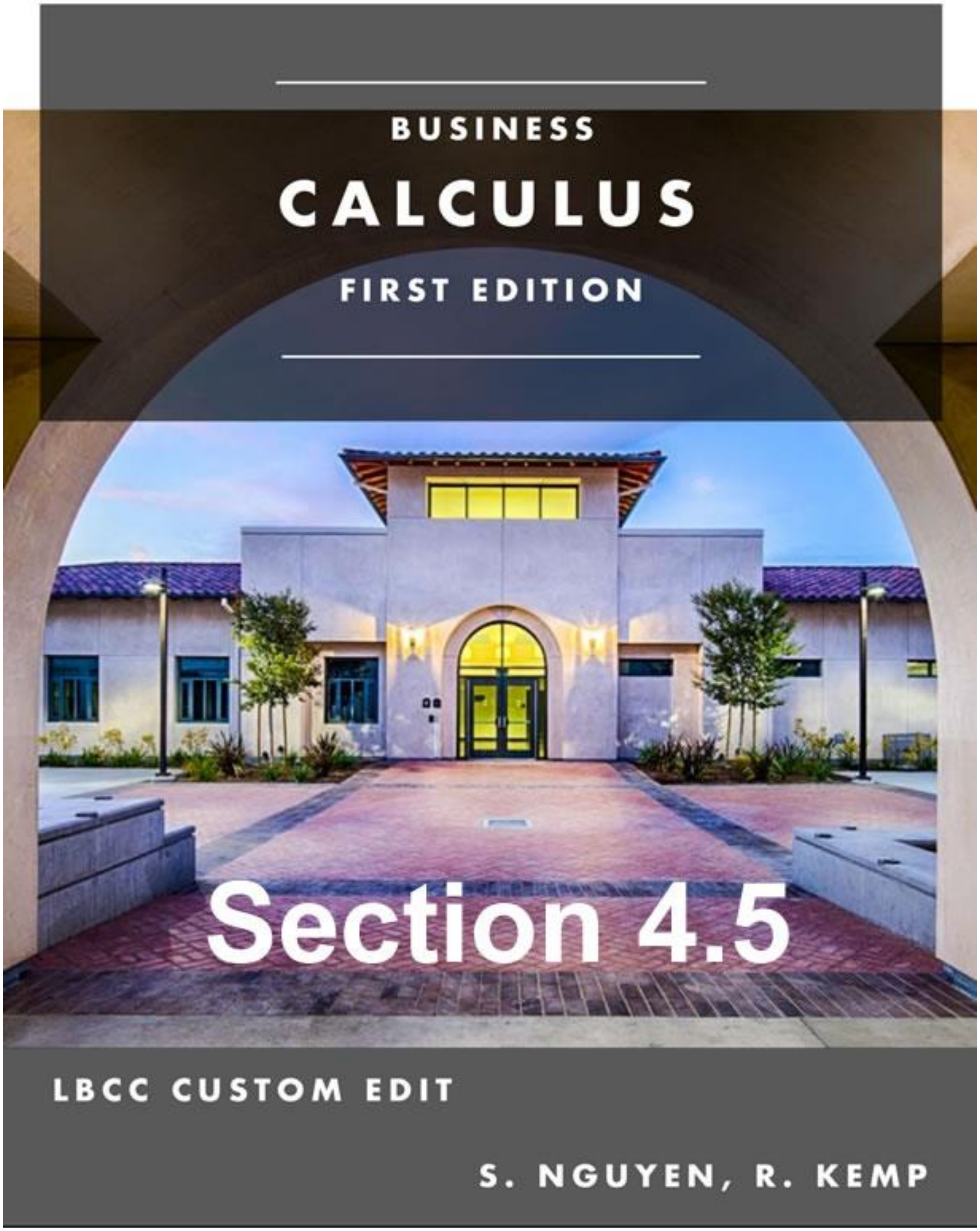
- $R(x, y) = -x^2 - y^2 + 30x + 25y$
- $P(x, y) = -x^2 - y^2 + 26x + 16y - xy - 20$
- $x = 12\ units\ of\ product - A, sell\ for\ \$18\ each$
- $y = 2\ units\ of\ product\ - B, sell\ for\ \$23\ each$
- $Rel. Max. value$: 152
- $Maximum\ Profit$: \$152

49.

- $R(x, y) = -2x^2 - 4y^2 + 50x + 100y$
- $C(x, y) = 10x + 12y + 100$
- $P(x, y) = -2x^2 - 4y^2 + 40x + 88y - 100$
- $x = 10\ slip - ons, sell\ for\ \$30\ each$
- $y = 11\ high - tops, sell\ for\ \$56\ each$
- $Rel. Max. value$: 584
- $Maximum\ Daily\ Profit$: \$584

50.

- $p_1 = \$55$
- $p_2 = \$65$
- $Maximum\ Revenue\ will\ be\ \$10{,}375$.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4.5 - LAGRANGE MULTIPLIERS - OPTIMIZATION

Introduction

Think back at some of the problems we solved in section 2.4, when we wanted to optimize functions of two variables subject to a given condition.

One such example was that we wanted to maximize the area of a rectangle, given that the perimeter was a fixed 40 ft.

The area of a reactangle is given by,

$$A = (length)(width) = xy,$$

And the perimeter of the rectangle is given by,

$$2(length) + 2(width) = 40,$$

$$2x + 2y = 40.$$

The solution to solving such a problem was to solve the equation given for the variable y, ($y = 20 - x$) and then substitute that y into the Area function, making it now a function of one variable only. $A = x(20 - x)$.

$$A = xy = x(20 - x)$$

This section addresses exactly these types of problems. We will want to optimize (minimize or maximize) a function that depends of several variables, given a certain condition (equation) that relates these variables.

We can still solve the problem by using the methods described in section 2.3 as above, but the problem is that sometimes the equation that relates these quantities is not easily solved for one of the variables, or the newly created function of one variable will have a derivative that is not easily found.

The French mathematician Joseph Louis Lagrange (1736–1813) invented a method, called the Method of Lagrange Multipliers, that would solve such problems using multivariable functions.

Lagrange Multipliers - Constraints & Objectives

The function that we are trying to minimize or maximize will be called the "**Objective**" function ($f(x, y)$, as it is the objective of the problem to optimize this function.

The "Objective Function" ($f(x, y)$) is always a multivariable function, such as is, for example the Rectangular Area function ($A_{Area}(x, y)$), as given below:

$$A_{Area}(x, y) = f(x, y) = xy$$

The condition that these variables have to satisfy will be transformed into a "**Constraint**" function ($g(x, y)$).

For example the condition above was that $2x + 2y = 40$. This "Constraint" equation would be now written as:

$$\big(g(x, y)\big) = 2x + 2y - 40 = 0$$

So, the above equation can be described as a "**Constraint**" equation, by moving all the terms to one side of the equation, and setting the equation equal to zero.

Lagrange next created a new function to combine the two, the objective and the constraint function, called the Lagrangian function:

$$L(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$$

The variable lambda (λ) introduced in this function is called the Lagrange multiplier. It's meaning is quite helpful as well.

It turns out that $|\lambda|$ is equal to the number of additional objective units for each additional constraint unit.

For example, in the area problem we would get $|\lambda|$ additional square feet of area for each additional foot added to the perimeter.

Now, we just have to optimize this new Lagrangian function $L(x, y, \lambda)$.

It turns out that the minimum or maximum of this function $L(x, y, \lambda)$ is the same as the min or max to the initial function $f(x, y)$ under the given condition.

We will not have to use the D-test, as in the previous section to see if a min or max occurs. If fact, some problems will give us several critical points, which will lead to both maximum and minimum values.

Lagrange's method turns out to be quite easy to follow. It eliminates the need to solve the constraint equation which, as mentioned above can sometimes be difficult to solve and we also don't need to use the complicated D-method to optimize a function of several variables.

All we need to do is find the critical points of the $L(x, y, \lambda)$ function and the minimum value and maximum values will occur at these points.

The proof for this method requires higher level calculus is not suited for this course. We will just this method here.

Next, we will discuss a process for optimizing, and therefore maximizing an "Objective" based on a particular "Constraint", using the Lagrange's Multipliers method.

Lagrange Multipliers

To Optimize:

The Objective Function ($f(x, y)$), subject to Constraint ($g(x, y) = 0$):

1. Write the Lagrangian Function, using the **Objective Function** ($f(x, y)$), and the **Constraint Function** ($g(x, y) = 0$)

$$L(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$$

2. Find the partials of the function ($L(x, y, \lambda)$), and set them equal to zero and solve.

$$L_x = 0 \quad , \quad L_y = 0 \quad , \quad L_\lambda = 0$$

3. The easiest way to solve the system of equations from above is to solve the first two equations $L_x = 0$ and $L_y = 0$ for (λ).

4. Set the two expressions for lambda (λ), in step 3, equal to create an equation between (x) and (y).

5. Combine the equation resulting from step 4 with the equation from $L_\lambda = 0$ and solve the system.

6. The solutions for (x) and (y): will create the critical points. The critical points are expressed: $CP(x, y)$.

7. Finally evaluate the **Objective Function** ($f(x, y)$), using the critical points: ($CP(x, y)$).

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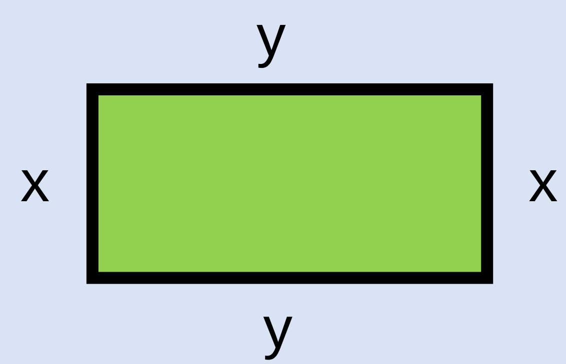
Example Problem #1

Maximizing the Area of an Enclosure

What is the maximum possible “area” of a rectangle with a “perimeter” of 40ft?



Solution:

Let (x) = width of the rectangle
(y) = length of the rectangle

The perimeter of a rectangle is the sum of all the sides, so

$$2x + 2y = 40$$

This is our "**Constraint**", or condition that must be satisfied by this rectangle.

And the “**Objective**” is to maximize is the Rectangular Area (A_{Area}).

The Objective Function:

$$f(x,y) = A_{Area}(x,y) = xy$$

Subject to Constraint Function:

$$g(x,y) = 2x + 2y - 40 = 0$$

Example Problem #1 – Cont'd

Maximizing the Area of an Enclosure

Next, we write a new function ($L(x, y, \lambda)$), called the **Lagrangian function**, which consists of the function to be maximized minus lambda (λ) times the constraint function:

$$L(x,y,\lambda) \;=\; f(x,y) \;-\; \lambda \cdot g(x,y)$$

$$L(x,y,\lambda) \;=\; xy \;-\; \lambda \cdot (2x \;+\; 2y \;-\; 40)$$

The Lagrangian function ($L(x, y, \lambda)$) is a function of three variables, x, y, and λ, and we begin as usual by setting its partials with respect to each variable equal to zero.

Find the partials of $L(x, y, \lambda)$ and set them equal to zero.

$$L_x \;=\; 0 \qquad , \qquad L_y \;=\; 0 \qquad , \qquad L_\lambda \;=\; 0$$

1. $L_x \;=\; y \;-\; 2\lambda \;=\; 0$
2. $L_y \;=\; x \;-\; 2\lambda \;=\; 0$
3. $L_\lambda \;=\; -(2x \;+\; 2y \;-\; 40\,) \;=\; 0$

(Note that the last equation is simply our Constraint.)

Next, solve for lambda (λ) in the two partials (L_x) and (L_y):

$$\lambda \;=\; \frac{y}{2} \qquad and \qquad \lambda \;=\; \frac{x}{2}$$

Example Problem #1 – Cont'd

Maximizing the Area of an Enclosure

Next, setting the two equation for lambda (λ) in the two partials (L_x) and (L_y), equal:

$$\frac{y}{2} = \frac{x}{2}$$

Solving the above yields:

$$y = x$$

Next, set ($y = x$) from the above solution, into the partial (L_λ), and solve:

$$L_\lambda = 2x + 2y - 40 = 0$$

$$L_\lambda = 2x + 2x - 40 = 0$$

$$4x = 40$$

$$x = 10$$

$$y = x = 10$$

The critical number found is ($CP(x, y)$) = (10, 10)

The Objective Function:

$$f(x, y) = A_{Area} = xy = (10\,ft)(10\,ft) = 100\,ft^2$$

The largest possible enclosure has width ($x = 10\ ft$) and length ($y = 10\ ft$)

Let's see the meaning of λ in this problem.

$$\lambda = \frac{x}{2} = 5$$

The max area of the rectangle would increase by 5ft^2 for each additional 1 ft added to the perimeter of the rectangle.

Example Problem #2

Maximizing the Surface Area of an Aluminum Can

An Energy Drink company wants to design a new can for their new product. The material to be used for the can is aluminum; and is cylindrical in shape. Design a can that requires the least amount of aluminum to be used; and can contain exactly 16 fluid ounces (28.875 cubic inches).

Find the radius and height of the can?

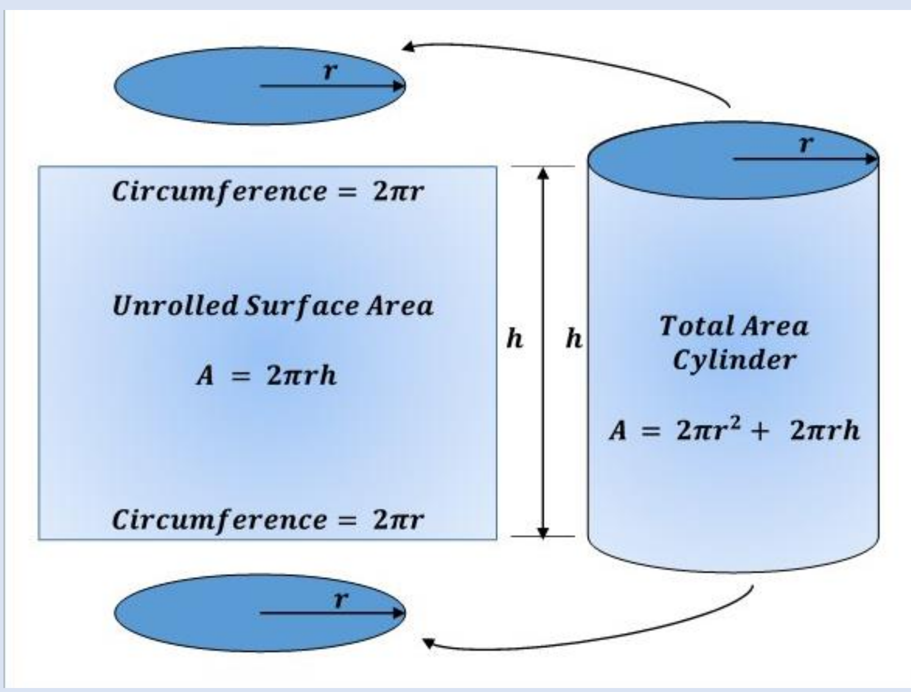
Solution:

For the aluminum, can, we want to minimize the amount of aluminum and the surface area of the cylindrical can; Let:

(r) = radius in inches of cylindrical can
(h) = height in inches of cylindrical can

The geometry of the cylindrical can is shown in the image below:

$$A_{Area} \;=\; (Top\,\&\,Bottom) + (Side) \;=\; 2\pi r^2 \;+\; 2\pi rh$$



$$Volume \;\rightarrow\; V_{ol} \;=\; \pi r^2 h \;=\; 28.875\; in^3$$

Example Problem #2 – Cont’d

Minimizing the Surface Area of an Aluminum Can

The Objective Function:

$$f(r,h) = A_{Area} = 2\pi r^2 + 2\pi rh$$

Subject to Constraint Function:

$$g(r,h) = \pi r^2 h - 28.875 = 0$$

Next, we write a new function ($F(r,h,\lambda)$), called the **Lagrange function**, which consists of the function to be maximized plus lambda (λ) times the constraint function:

$$L(r,h,\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

$$L(r,h,\lambda) = 2\pi r^2 + 2\pi rh - \lambda \cdot (\pi r^2 h - 28.875)$$

The Lagrange function ($L(r,h,\lambda)$) is a function of three variables, r, h, and λ, and we begin as usual by setting its partials with respect to each variable equal to zero.

Set the partials of $L(r,h,\lambda)$, equal to zero.

$$L_r = 0 \quad , \quad L_h = 0 \quad , \quad L_\lambda = 0$$

1. $L_r = 4\pi r + 2\pi h - 2\pi rh\lambda = 0$
2. $L_h = 2\pi r - \pi r^2 \lambda = 0$
3. $L_\lambda = -(\pi r^2 h - 28.875\,) = 0$

Next, solve for lambda (λ) in the two partials (L_r) and (L_h):

$$\lambda = \left(\frac{4\pi r + 2\pi h}{2\pi rh}\right) = \left(\frac{2r + h}{rh}\right)$$

$$\lambda = \left(\frac{2\pi r}{\pi r^2}\right) = \left(\frac{2}{r}\right)$$

Example Problem #2 – Cont'd

Next, setting the two equation for lambda (λ) in the two partials (L_r) and (L_h), equal:

$$\lambda = \left(\frac{2r + h}{rh}\right) = \left(\frac{2}{r}\right)$$

Solving the above yield:

$$h = 2r$$

Next, set either ($r = \frac{h}{2}$) or ($h = 2r$) from the above solution, into the partial (L), and solve:

$$L_\lambda = -(\pi r^2 h - 28.875) = 0$$

$$\pi r^2(2r) - 28.875 = 0$$

$$2\pi r^3 = 28.875$$

Where the Critical Values are:

$$r = \sqrt[3]{\frac{28.875}{2\pi}}\ in = \sqrt[3]{4.60} = 1.66\ inches$$

$$h = 2r = 3.325\ inches$$

Therefore, the most economical 16-fluid-ounce aluminum cylindrical can has a radius of ($r = 1.66\ inches$) and a height of ($h = 3.325\ inches$).

Let's see the meaning of λ in this problem.

$$\lambda = \left(\frac{2}{r}\right) = \left(\frac{2}{1.66}\right) \approx 1.21$$

The max "Area" of the 16-fluid-ounce aluminum cylindrical would increase by approximately $1.21ft^2$ for each additional 1 cubic inches added to the volume of the cylincrical can.

Example Problem #3

Maximizing the Production Rate

A company's production output, is given by the Cobb–Douglas production function ($P(L,K) = 720L^{\frac{1}{2}}K^{\frac{1}{2}}$), where ($L$) and ($K$) are the numbers of units of labor and capital that are used.

To stay within its budget of $6480, these amounts must satisfy

$$20L + 90K = 6480.$$

Find the amounts of labor and capital that maximize production.

What is the maximum number of items that can be produced?

Solution:

To Maximize a company's production; Let:

(L) – Units of Labor
(K) – Units of Production

The Objective Function:

$$f(L,K) = P(L,K) = 720L^{\frac{1}{2}}K^{\frac{1}{2}}$$

Subject to Constraint Function:

$$g(L,K) = 20L + 90K - 6480 = 0$$

Next, we write a new function ($L(L,K,\lambda)$), called the **Lagrangian function**, which consists of the function to be maximized minus lambda (λ) times the constraint function:

$$L(L,K,\lambda) = f(L,K) - \lambda \cdot g(L,K)$$

$$L(L,K,\lambda) = 720L^{\frac{1}{2}}K^{\frac{1}{2}} - \lambda \cdot (20L + 90K - 6480)$$

Example Problem #3 – Cont'd

The Lagrange function ($L(L, K, \lambda)$) is a function of three variables, L, K, and λ, and we begin as usual by setting its partials with respect to each variable equal to zero.

Set the partials of $L(L, K, \lambda)$, equal to zero.

$$L_L = 0 \quad , \quad L_K = 0 \quad , \quad L_\lambda = 0$$

1. $L_L = 360L^{\frac{-1}{2}}K^{\frac{1}{2}} - 20\lambda = 0$

2. $L_K = 360L^{\frac{1}{2}}K^{\frac{-1}{2}} - 90\lambda = 0$

3. $L_\lambda = -(20L + 90K - 6480) = 0$

Next, solve for lambda (λ) in the two partials (L_L) and (L_K):

$$\lambda = \left(\frac{360L^{\frac{-1}{2}}K^{\frac{1}{2}}}{20}\right) = \left(18L^{\frac{-1}{2}}K^{\frac{1}{2}}\right) = 18\left[\frac{K}{L}\right]^{\frac{1}{2}}$$

$$\lambda = \left(\frac{360L^{\frac{1}{2}}K^{\frac{-1}{2}}}{90}\right) = \left(4L^{\frac{1}{2}}K^{\frac{-1}{2}}\right) = 4\left[\frac{L}{K}\right]^{\frac{1}{2}}$$

Next, setting the two equation for lambda (λ) in the two partials (L_L) and (L_K), equal:

$$\lambda = 18\left[\frac{K}{L}\right]^{\frac{1}{2}} = 4\left[\frac{L}{K}\right]^{\frac{1}{2}}$$

Solving the above yield:

$$4L = 18K$$

Example Problem #3 – Cont'd

Next, set either ($K = \frac{2L}{9}$) or ($L = \frac{9k}{2}$) from the above solution, into the partial (L_λ), and solve:

$$L_\lambda = -(20L + 90K - 6480) = 0$$

$$20\left(\frac{9k}{2}\right) + 90K - 6480 = 0$$

$$180K = 6480$$

Where the Critical Values are:

$$K = \frac{6480}{180} = 36 \quad units\ of\ Capital$$

$$L = \frac{9k}{2} = 162 \quad units\ of\ Labor$$

The company should use ($L = 162$) units of Labor, and ($K = 36$) units of Capital.

The number of items that can be produced is:

$$P(L, K) = 720L^{\frac{1}{2}}K^{\frac{1}{2}}$$

$$P(216, 24) = 720(162)^{\frac{1}{2}}(36)^{\frac{1}{2}} = 54{,}984.62$$

$$P(L, K) = 54{,}984.62 \quad units\ of\ Production$$

The company can produce a maximum of ($54{,}984$) items using 162 units of labour and 36 units of capital.

Meaning of λ in this problem.

$$\lambda = 18\left[\frac{K}{L}\right]^{\frac{1}{2}} = 18\left[\frac{36}{162}\right]^{\frac{1}{2}} \approx 8.5$$

The company's production output will increase by 8.5 units for each additional dollar added to the budget constraint of \$6480.

Example Problem #4

Maximizing & Minimizing Two Extreme Values

Sometimes we can both Maximize & Minimize the Objective function.

Find both the Min and the Max of the function:

$$f(x,y) = x^2 + y^2$$

Subject to the Constraint function:

$$xy = 1$$

$$g(x,y) = xy - 1 = 0$$

Solution:

First, we write the Lagrangian function $L(x,y,\lambda)$:

$$L(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

$$L(x,y,\lambda) = x^2 + y^2 - \lambda \cdot (xy - 1)$$

Find and set the partials of $L(x,y,\lambda)$, equal to zero.

$$L_x = 0 \quad , \quad L_y = 0 \quad , \quad L_\lambda = 0$$

1. $L_x = 2x - y\lambda = 0$
2. $L_y = 2y - x\lambda = 0$
3. $L_\lambda = (xy - 1) = 0$

Example Problem #4 – Cont'd

Next, solve for lambda (λ) in the two partials (L_x) and (L_y):

$$\lambda = \frac{2x}{y}$$

$$\lambda = \frac{2y}{x}$$

Next, setting the two equation for lambda (λ) equal:

$$\lambda = \left(\frac{2y}{x}\right) = \left(\frac{2x}{y}\right)$$

Solving the above yield:

$$2y^2 = 2x^2$$

$$y = \pm x$$

Thus, there are two values for (y).

$$\begin{cases} y_1 = x \\ y_2 = -x \end{cases}$$

Next, from the above solution, substitute into the partial (l_λ), and solve:

$$L_\lambda = xy - 1 = 0$$

$$L_\lambda = x^2 - 1 = 0 \quad \text{or} \quad L_\lambda = -x^2 - 1 = 0$$

$$x^2 - 1 = 0 \quad \text{or} \quad -x^2 - 1 = 0$$

$$x = \pm 1 \qquad\qquad x^2 = -1$$

impossible

Critical Values:

$x = \pm 1$ and x = y, so the two critical points are (1, 1) and (-1, -1)

Example Problem #4 – Cont'd

We evaluate the objective function ($f(x, y) = x^2 + y^2$) at each of the critical points.

Critical Point	$f(x, y) = x^2 + y^2$	Max & Min
$(1, 1)$	2	Maximum
$(-1, -1)$	2	Maximum

In this case both critical point lead to a the same λ - value.

Let's see the meaning of λ in this problem.

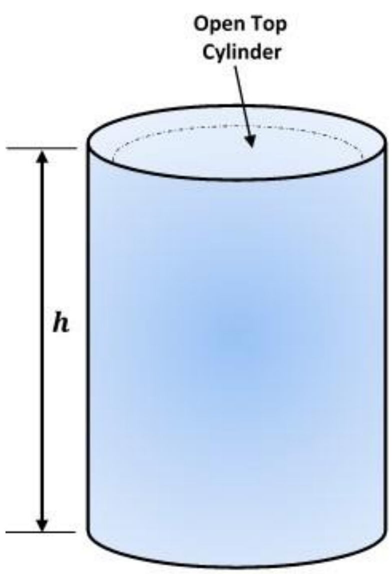
$$\lambda = 2\left(\frac{y}{x}\right) = 2\left(\frac{1}{1}\right) = 2\left(\frac{-1}{-1}\right) = 2$$

The max Objective would increase by 2 units for each additional 1 unit of "Constraint" added to the to overall "Objective".

4.5 - EXERCISES

Use Lagrange Multipliers to Maximize each function ($f(x,y)$) subject to the constraint (The Maximum values do exist.)			
1.	$f(x,y) = 2xy$ and $3x + 2y = 36$	2.	$f(x,y) = 49 - x^2 - y^2$ and $x + 3y = 10$
Use Lagrange Multipliers to Minimize each function ($f(x,y)$) subject to the constraint (The Minimum values do exist.)			
3.	$f(x,y) = ln(x^2 + y^2)$ and $x + 2y = 48$	4.	$f(x,y) = e^{(x^2 + y^2)}$ and $\frac{1}{3}x + y = 1$
5.	$f(x,y) = x^2 + y^2$ and $x + 2y = 25$		

Use Lagrange Multipliers to Minimize and maximize each function ($f(x,y)$) subject to the constraint			
6.	$f(x,y) = xy$ and $x^2 + y^2 = 50$	7.	$f(x,y) = 3x + 4y$ and $x^2 + y^2 = 1$

8.	A firm is constructing an open-top box with a square base and a volume of 108 in^3. What dimensions yield the minimum surface area? What is that area?
9.	A company wants to build an Oil container, that is cylindrical in shape with an open top; and the cylinder is to be made, with the least amount of material. The total area of the cylinder is bottom area plus the side area. Find the dimensions of the cylinder if the volume is to be: Volume = 240 cubic feet 
10.	A company's production output, is given by the Cobb–Douglas production function ($P(L,K) = 600L^{\frac{1}{4}}K^{\frac{3}{4}}$), where ($L$) and ($K$) are the numbers of units of labor and capital that are used. To stay within its budget of \$4800, these amounts must satisfy $$60L + 40K = 4800.$$ Find the amounts of labor and capital that maximize production. What is the maximum number of items that can be produced? Find and give an interpretation of the variable λ.

Solutions

1. $Maximum: \quad f(x,y) = 108 \quad at \quad x = 6, \quad y = 9$

2. $Maximum: \quad f(x,y) = 39 \quad at \quad x = 1, \quad y = 3$

3. $Maximum: \; f(x,y) = \ln(460.8) = 6.13 \; at \; x = 9.6, \; y = 19.2$

4. $Maximum: \quad f(x,y) = e^{0.9} \quad at \quad x = 0.3, \quad y = 0.9$

5. $Maximum: \quad f(x,y) = 125 \quad at \quad x = 5, \quad y = 10$

6. $Maximum\ f(x,y) = 25\ at\ x = 5, y = 5, and\ x = -5, y = -5$
   $Minimum\ f(x,y) = -25\ at\ x = 5, y = -5, and\ x = -5, y = 5$

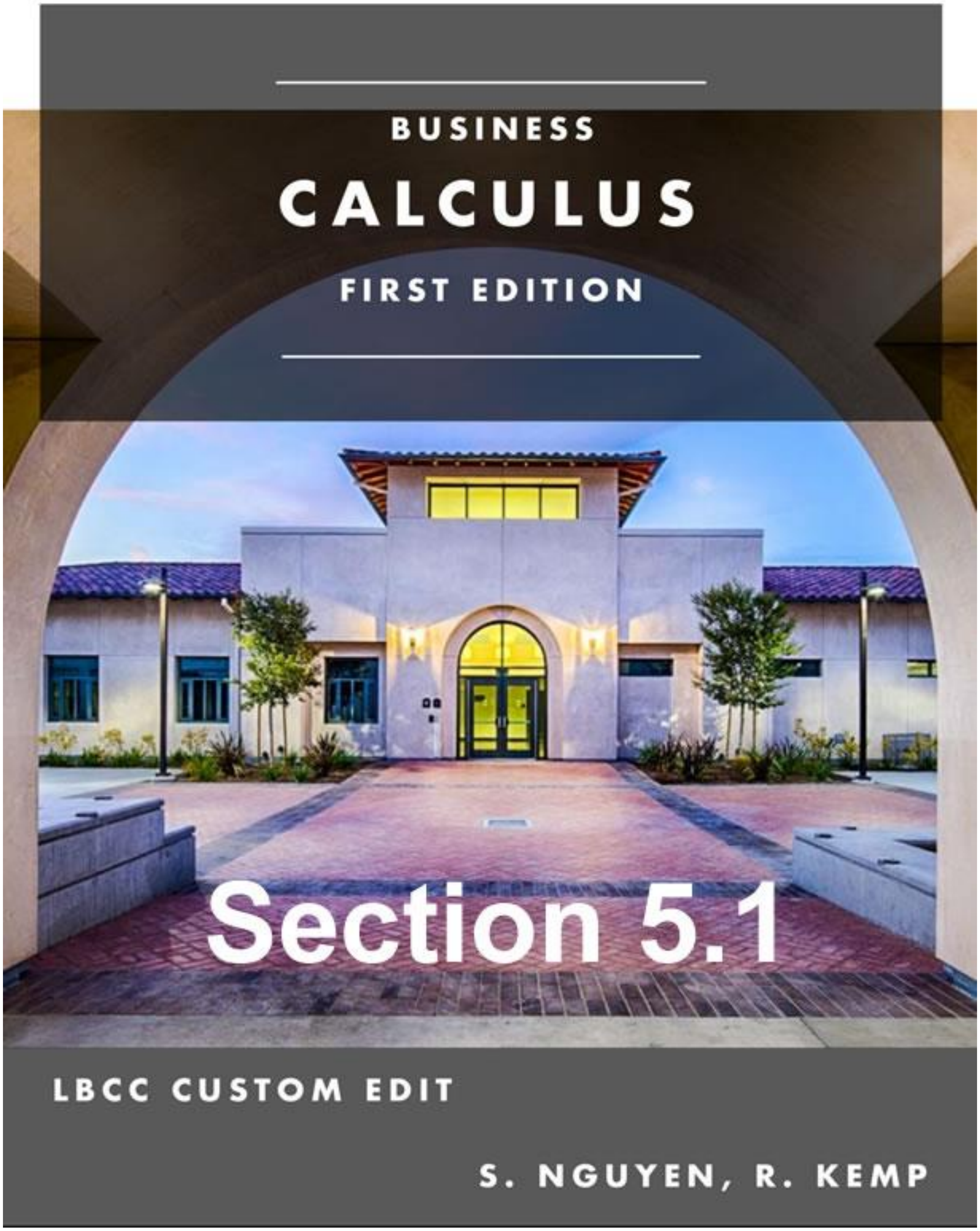
7. $Maximum\ f(x,y) = 5\ at\ x = 3/5, y = 4/5$
   $Minimum\ f(x,y) = -5\ at\ x = -3/5, y = -4/5$

8.
   . $The\ base\ is\ 6\ in\ by\ 6\ in\ \ and\ the\ height\ is\ 3\ in$

   a. $Surface\ area\ is\ 108in^2$

9. $r \approx 4.24\ feet, \quad h = 4.24\ feet$

10. $L = 20\ units, K = 90\ units, P(20,90) \approx 65{,}931\ units$
    $\lambda \approx 7.7$


BUSINESS
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5.1 - INDEFINITE INTEGRATION

Introduction

The previous chapters dealt with **Differential Calculus**, studying differentiation and its uses. We started with the simple geometrical idea of the slope of a tangent line to a curve, developed it into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives.

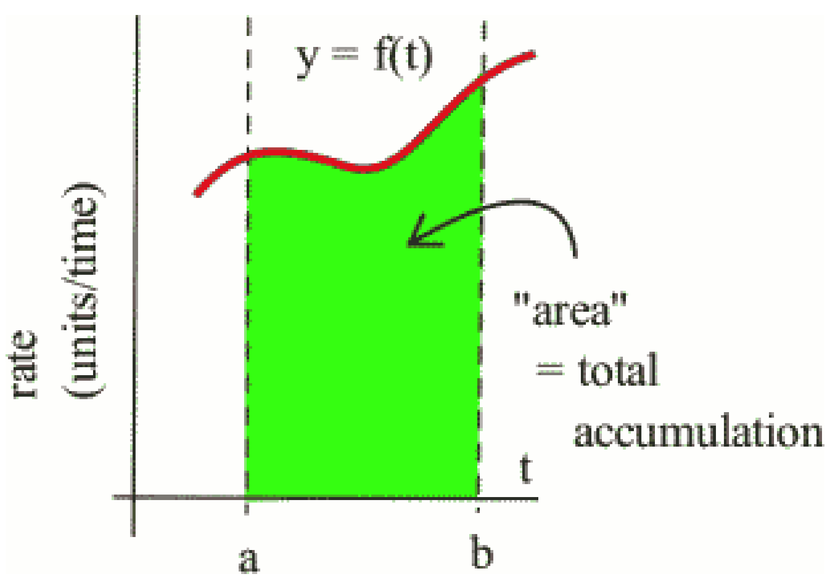
This chapter deals with the concepts of **Integral Calculus** also known as **Antidifferentiation** and starts with the simple geometric idea of area.

Similarly, the concepts of **Integral Calculus** will be developed into a combination of theory, techniques, and applications.

The work of Integration calculus, is the reverse process of taking a derivative, also known as antidifferentiation; which, for a given derivative, restores the original function.

We use the symbol ($\int$) to denote an integration process. It will be clear in a later section why the integration symbol resembles the letter S.

We will be using integration for purposes other than restoring a function, given its derivative, such as finding "areas" beneath or between curves, finding the average value a function can take or applications such as consumer's and producer's surplus.



Indefinite Integration – The Antiderivative

This section deals with the Calculus concepts of the **Indefinite Integral**, a mathematical process also known as **Antidifferentiation**.

The generalized work of Integration calculus, is the reverse process of taking a derivative, also known as antidifferentiation; which, for a given derivative, restores the original function.

Antidifferentiation has many business applications; For example, a "differentiation" turns a **cost function** ($C(x)$) into a **marginal cost function** ($MC(x) = \frac{dC(x)}{dx}$), and "antidifferentiation" also known as "integration", turns a **marginal cost function** ($MC(x)$) back into a **cost function** ($C(x)$).

$$C(x) = \int MC(x)\, dx$$

We use the symbol ($\int$) to denote an integration process. The antiderivative is also called the **indefinite integral**.

Indefinite Integral

$$\int f(x)\, dx = F(x) + K$$

Where

$$F'(x) = \frac{d}{dx}F(x) = f(x)$$

The ($\int$) symbol is called an **integral sign**; the (dx) defines the variable (x), by which the function changes; the function ($f(x)$) is called the **integrand**.

The integral of a function ($f(x)$), results in an output function ($F(x)$), where ($F'(x) = f(x)$).

An antiderivative or the integral of a function ($f(x)$), is a whole family of functions, written ($y(x) = F(x) + K$), where (K) represents any arbitrary fixed constant.

Power Rule for Integration

Remember that, the **reverse** process of **differentiation** is **Integration**. Thus, calculus provides several "rules" that allow us to simplify the integration process.

Same as one of the most important rules of differentiation was the Power Rule, one of the most used rules in "Integral Calculus", is the "Power Rule for Integration".

Power Rule for Integration

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + K$$

Where: ($n \neq -1$)

The "Power Rule for Integration" simply states, to integrate (x^n), add the number one (1) to the original power (x^{n+1}), and divide by the new power ($n + 1$), then add the arbitrary constant (K).

The above "Power Rule for Integration" works for any positive or negative power, besides ($n = -1$), which clearly does not work, since the denominator of the fraction would become equal to zero (0).

Remembering that, an "integration" is the reverse of a "differentiation", we can always check our answer, by differentiating the above result, to see if we get back to the original integrand (x^n).

If the output function is an antiderivative ($F(x) = x^2$), with a single variable (x), then ($f(x)$) has an infinite number of antiderivatives, and every one of them has the form ($y(x) = F(x) + K$).

In fact, there are infinitely many functions ($y(x) = F(x) + K$), that all have the same derivative.

We could shift the graph of function ($y(x) = F(x) + K$) up or down, based on the fixed constant (K), and the shape of the curve would not be affected, thus the derivative would be the same.

The derivative ($F'(x)$) of the function – ($F(x) = x^2$); is given by the following:

$$F'(x) = \frac{d}{dx}(x^2) = f(x) = 2x$$

The antiderivative or the integration, of the function ($f(x) = 2x$), is also given by the following:

$$\int f(x)\, dx = F(x) + K$$

$$y(x) = \int 2x\, dx = x^2 + K$$

Since, the arbitrary constant (K), represents any non-changing or fixed value positive, negative, or zero, the integral of ($f(x) = 2x$), yields an infinite number of antiderivatives.

Example Problem #1

Find the Indefinite Integral of input function ($f(x) = 2x$)?

Solution:

$$\int f(x)\,dx = F(x) + K$$

$$y(x) = \int 2x\,dx = x^2 + 5$$

$$y(x) = \int 2x\,dx = x^2 - 47.345$$

$$y(x) = \int 2x\,dx = x^2 + \pi$$

Where the arbitrary constant is: ($K = 5,\ \ or\ \ K = -47.345,\ \ or\ \ K = \pi$)

In fact, there are lots of answers, there are infinitely many functions that all have the same derivative.

Therefore, the graph of the output function ($F(x) = x^2$) tells us about the shape of the function, and the arbitrary constant (K), tells us about the location of the function relative to a coordinate system.

Clearly, we can add any arbitrary constant (K), to the antiderivative output function ($F(x) = x^2 + K$), and the derivative will still be ($F'(x) = f(x) = 2x$)

$$f(x) = F'(x) = \frac{d}{dx}(x^2 + 5) = 2x$$

$$f(x) = F'(x) = \frac{d}{dx}(x^2 - 47.345) = 2x$$

We could shift the graph of the function by the arbitrary constant (K) up or down and the shape ($F(x) = x^2$) wouldn't be affected, so the derivative would be the same; ($f(x) = F'(x)$).

Example Problem #2

Perform the following Indefinite Integral on the following function:

$$y(u) = \int u^4 \, du$$

Solution:

$$y(u) = \frac{u^{4+1}}{4+1} + K = \frac{1}{5}u^5 + K$$

Example Problem #3

Perform the following Indefinite Integral on the following function:

$$y(t) = \int \frac{1}{t^3} \, dt$$

Solution:

$$y(t) = \int t^{-3} \, dt = \frac{t^{-3+1}}{-3+1} + K$$

$$y(t) = -\frac{1}{2}t^{-2} + K = -\frac{1}{2t^2} + K$$

Example Problem #4

Perform the following Indefinite Integral on the following “single variable” function:

$$y(x) = \int x \, dx$$

Solution:

$$y(x) = \int x \, dx = \frac{x^{1+1}}{1+1} + K = \frac{1}{2}x^2 + K$$

Exception to the Power Rule for Integration – ($n = -1$)

There is an exception to the Power Rule for Integration, when the power or exponent is equal to negative one ($n = -1$); the integration fails or becomes undefined ($\frac{1}{0}$) and leads to the expression.

$$\int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} + K = \frac{x^0}{0} + K$$

$$\int x^{-1}\,dx = \frac{1}{0} + K = undefined$$

For this reason, there is an exception to the Power Rule for Integration when the power or exponent equals negative one ($n = -1$).

Remember, that the differentiation formula for the natural logarithm of the function of ($f(x) = ln\,x$), is given by the following.

$$\frac{d}{dx} ln\,(x) = \frac{d}{dx} ln(-x) = \frac{1}{x}$$

Therefore, since integration is the reverse of differentiation.

$$\int \frac{1}{x}\,dx = ln|x| + K$$

Thus, the following rule applies when the power or exponent of the function (x^n) equals negative one ($n = -1$).

Exception to Power Rule for Integration

$$\int x^n\,dx = \int x^{-1}\,dx = \int \frac{1}{x}\,dx = ln|x| + K$$

Where: ($n = -1$)

The **logarithmic function** result above, is limited in the Domain: $(0, \infty)$ and the Range: $(-\infty, \infty)$. Since only positive values along the domain of (x) can be found inside a logarithmic function, we need the absolute value $|x|$symbol around the x.

The "Rules for Indefinite Integration"

Rules for Indefinite Integration

Constant Multiple Integration Rule

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)dx$$

Integration of a Constant Rule

$$\int k\,dx = kx + K$$

Integration Sum & Difference Rule

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Integration Power Rule

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + K & if \quad n \neq -1 \\ ln|x| + K & if \quad n = -1 \end{cases}$$

Example Problem #5

Perform the following Indefinite Integral on the following "Constant" function:

$$y(x) = \int 5\,dx$$

Solution:

$$y(x) = \int 5\,dx = 5\int dx = 5x + K$$

Example Problem #6

Perform the following Indefinite Integral using the "Constant Multiple" rule:

$$y(x) = \int 12\, x^3\, dx$$

Solution:

$$y(x) = \int 12\, x^3\, dx = 12 \int x^3\, dx$$

$$y(x) = 12 \left[\frac{x^{3+1}}{3+1}\right] + K = \frac{12}{4}\, x^4 + K$$

$$y(x) = 3x^4 + K$$

Example Problem #7

Perform the following Indefinite Integral using the "Sum" rule:

$$y(x) = \int (x^2 + x^5)\, dx$$

Solution:

$$y(x) = \int x^2\, dx + \int x^5\, dx$$

$$y(x) = \left[\frac{x^{2+1}}{2+1}\right] + K_1 + \left[\frac{x^{5+1}}{5+1}\right] + K_2$$

$$K = K_1 + K_2$$

$$y(x) = \frac{1}{3}\, x^3 + \frac{1}{6}\, x^6 + K$$

Example Problem #8

Perform the following Indefinite Integral "Difference" rule:

$$y(x) = \int \left(3x^7 - \frac{14}{x^2} - \frac{15}{\sqrt{x}}\right) dx$$

Solution:

$$y(x) = 3\int x^7 dx - 14\int x^{-2}\, dx - 15\int x^{-\frac{1}{2}}\, dx$$

$$y(x) = 3\left[\frac{x^{7+1}}{7+1}\right] + K_1 - 14\left[\frac{x^{-2+1}}{-2+1}\right] + K_2 - 15\left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + K_3$$

$$K = K_1 + K_2 + K_3$$

$$y(x) = \frac{3}{8}x^8 - 14\left[\frac{x^{-1}}{-1}\right] - 15\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right] + K$$

$$y(x) = \frac{3x^8}{8} + 14\,x^{-1} - 30\,x^{\frac{1}{2}} + K$$

$$y(x) = \frac{3x^8}{8} + \frac{14}{x} - 30\sqrt{x} + K$$

Example Problem #9

Perform the following Indefinite Integral on "**Polynomial**" functions. What do you notice?

1. $$y(x) = \int (4x^3)\,dx = \frac{4}{4}x^4 + K = x^4 + K$$

2. $$y(x) = \int (5x^4)\,dx = \frac{5}{5}x^5 + K = x^5 + K$$

3. $$y(x) = \int (6x^5)\,dx = \frac{6}{6}x^6 + K = x^6 + K$$

Example Problem #10

Perform the following Indefinite Integral on "**Negative Exponent**" functions. What do you notice?

1. $$y(x) = \int (2x^{-3})\,dx = \frac{2}{-2}x^{-2} + K$$
$$y(x) = -x^{-2} + K = -\frac{1}{x^2} + K$$

2. $$y(x) = \int (3x^{-4})\,dx = \frac{3}{-3}x^{-3} + K$$
$$y(x) = -x^{-3} + K = -\frac{1}{x^3} + K$$

3. $$y(x) = \int (4x^{-5})\,dx = \frac{4}{-4}x^{-4} + K$$
$$y(x) = -x^{-4} + K = -\frac{1}{x^4} + K$$

Example Problem #11

Perform the following Indefinite Integral on the "**Root**" functions. What do you notice?

1. $y(x) = \int \left(4\sqrt[3]{x}\right) dx = 4\int \left(x^{\frac{1}{3}}\right) dx = 4\left[\frac{x^{\left(\frac{1}{3}+1\right)}}{\frac{1}{3}+1}\right] + K$

$$y(x) = \frac{4}{\left(\frac{4}{3}\right)} x^{\frac{4}{3}} + K = 3x^{\frac{4}{3}} + K$$

2. $y(x) = \int \left(5\sqrt[4]{x}\right) dx = 5\int \left(x^{\frac{1}{4}}\right) dx = 5\left[\frac{x^{\left(\frac{1}{4}+1\right)}}{\frac{1}{4}+1}\right] + K$

$$y(x) = \frac{5}{\left(\frac{5}{4}\right)} x^{\frac{5}{4}} + K = 4x^{\frac{5}{4}} + K$$

3. $y(x) = \int \left(6\sqrt[5]{x}\right) dx = 6\int \left(x^{\frac{1}{5}}\right) dx = 5\left[\frac{x^{\left(\frac{1}{4}+1\right)}}{\frac{1}{4}+1}\right] + K$

$$y(x) = \frac{6}{\left(\frac{6}{5}\right)} x^{\frac{6}{5}} + K = 5x^{\frac{6}{5}} + K$$

Example Problem #12

Perform the following Indefinite Integral:

$$y(x) = \int (28x^3 - 3x^{-2} + 9 - x)\,dx$$

Solution:

$$y(x) = 28\int x^3\,dx - 3\int x^{-2}\,dx + 9\int 1\,dx - \int x\,dx$$

$$y(x) = 28\left[\frac{x^{3+1}}{3+1}\right] - 3\left[\frac{x^{-2+1}}{-2+1}\right] + 9\left[\frac{x^{0+1}}{0+1}\right] - \left[\frac{x^{1+1}}{1+1}\right] + K$$

$$y(x) = 28\left[\frac{x^4}{4}\right] - 3\left[\frac{x^{-1}}{-1}\right] + 9x - \left[\frac{x^2}{2}\right] + K$$

$$y(x) = 7x^4 + \frac{3}{x} + 9x - \frac{1}{2}x^2 + K$$

Example Problem #13

Perform the following Indefinite Integral:

$$y(x) = \int x^3(x+4)^2\,dx$$

Solution:

Since we do not have a Rule for the product of two functions, we will need to distribute first.

$$y(x) = \int x^3(x+4)^2\,dx = \int x^3(x^2 + 8x + 16)\,dx$$

$$y(x) = \int (x^5 + 8x^4 + 16x^3)\,dx$$

$$y(x) = \left[\frac{x^{5+1}}{5+1}\right] + 8\left[\frac{x^{4+1}}{4+1}\right] + 16\left[\frac{x^{3+1}}{3+1}\right] + K$$

$$y(x) = \frac{1}{6}x^6 + \frac{8}{5}x^5 + 4x^4 + K$$

Business Cases using Indefinite Integration

Now, let's consider a couple of business case examples. If you remember, a "differentiation" turns a cost function ($C(x)$), into a marginal cost function ($MC(x) = \frac{dC(x)}{dx}$), and "antidifferentiation" also known as "integration", turns the marginal cost function back into a cost function.

$$C(x) = \int MC(x)\, dx$$

Example Problem #14

A company's **marginal cost** function is ($MC(x)$), for producing an x amount of units is given below. The fixed costs are \$1200.

Find the Cost function ($C(x)$).

$$MC(x) \;=\; 10\sqrt[3]{x^2}$$

Solution:

$$C(x) \;=\; \int MC(x)\, dx \;=\; 10\int x^{\frac{2}{3}}\, dx$$

$$C(x) \;=\; 10\left[\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right] + K \;=\; 10\left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\right] + K$$

$$C(x) \;=\; 6x^{\frac{5}{3}} + K \;=\; 6\sqrt[3]{x^5} + K$$

The fixed cost is used to determine a value for the constant K. Since the fixed cost is the amount of money the company had to spend before producing any items, i.e. when 0 items were produced: $x = 0$, $C(0) = 1200$.

$$C(0) \;=\; 6\left(0^{\frac{5}{3}}\right) + K \;=\; K = 1200$$

Finally, the Cost function ($C(x)$) is found to be:

$$C(x) \;=\; 6\sqrt[3]{x^5} + 1200$$

Example Problem #15

Assume the annual **rate of change** in the national debt of a country (in billions of dollars per year) can be modeled by the function

$$D'(t) = 635 + 128t - 186t^2 + 12t^3,$$

where t is years since 2000.

a) Consider that the national debt was $2,000 billion in the year 2000, find a formula that can be used to compute the national debt of this country t years after the year 2000.

b) Use this function to find the national debt in the year 2020.

Solution:

$$D(t) = \int D'(t)\,dt = \int (635 + 128t - 186t^2 + 12t^3)\,dt$$

$$D(t) = 635t + 128\left(\frac{t^{1+1}}{1+1}\right) - 186\left(\frac{t^{2+1}}{2+1}\right) + 12\left(\frac{t^{3+1}}{3+1}\right) + K$$

$$D(t) = 635t + 64t^2 - 62t^3 + 3t^4 + K$$

Next, we need to determine the value of the arbitrary constant (K).

The debt function evaluated at ($t = 0$), makes the constant (K), equal to the initial debt ($D(0) = \$2000 \text{ Billion dollars}$).

$$D(0) = 635(0) + 64(0) - 62(0) + 3(0) + K = 2000$$

$$K = 2000$$

Finally, the $D(t)$, is completed.

$$D(t) = 635t + 64t^2 - 62t^3 + 3t^4 + 2000$$

Next, use this function to evaluate the debt in year 2020, after ($t = 20\ years$).

$$D(20) = 24{,}300$$

The national debt in year 2020 will be $24,300 Billion Dollars.

Summary

In this section, we have discussed various techniques of Indefinite Integration.

Rules for Indefinite Integration

Constant Multiple Integration Rule - $\int k \cdot f(x)\,dx \;=\; k \cdot \int f(x)dx$

Integration of a Constant Rule - $\int k\,dx \;=\; kx \;+\; K$

Integration Sum - & Difference Rule $\int [f(x) \pm\ g(x)]\,dx \;=\; \int f(x)\,dx \;\pm\; \int g(x)\,dx$

Integration Power Rule - $\int x^n\,dx \;=\; \begin{cases} \dfrac{x^{n+1}}{n+1} \;+\; K & if \quad n \neq -1 \\ ln|x| \;+\; K & if \quad n = -1 \end{cases}$

Where we defined the process of integration as being the reverse of the process of differentiation; and allows you to recover any quantity from its rate of change.

The business case for integration allows you to recover the profit, revenue, and cost functions, given the marginal profit, revenue, and cost functions.

5.1 - EXERCISES

Find each indefinite Integral			
1.	$\int x^2\,dx$	2.	$\int \frac{1}{x^2}\,dx$
3.	$\int \sqrt[3]{x}dx$	4.	$\int (3x^2 + 5x - 7\,)\,dx$
5.	$\int 12.3\,dx$	6.	$\int (2.4x^5 - x - 1.25)\,dx$
7.	$\int \left(\sqrt{x} - \frac{1}{4x^3}\right)dx$	8.	$\int \left(11\sqrt[6]{x^5} - \frac{1}{5\sqrt[4]{x^3}}\right)dx$
9.	$\int (x+2)(x-2)dx$	10.	$\int \left(\frac{x^5 - x^2}{x}\right)dx$
11.	$\int -5x^3(14x^3 - x^{-3})dx$	12.	$\int \frac{3x^2 - 13x - 10}{3x + 2}dx$
13.	A company determines their Marginal Cost of production in dollars per item, is $(MC(x))$, where (x) is the number of units, and their fixed costs are \$4000.00. Find the Cost function? $MC(x) = \frac{4}{\sqrt{x}} + 2$		

14.	A company determines their Marginal Cost function, in dollars per item is $\left(MC(x) = 2x^2 - \frac{4}{\sqrt[3]{x}} + 10\right)$, and that their Marginal Revenue function is $\left(MR(x) = 4x^3 - x^2\right)$ where (x) is the number of units produced and sold. Find the **Profit function** knowing that the fixed costs are $1200; and when 0 items are sold, the revenue is equal to $0. **Hint:** After integrating consider when the Revenue: $R(0) = 0,\ \ x = 0$, and the Total Cost: $C(0) = C(Variable) + C(Fixed)$
15.	A robot has been programmed so that when it starts to move, its velocity after *t* seconds will be $v(t)$ feet/second. (a) How far will the robot travel during its first t seconds of movement? (b) Use the formula to find how far will the robot travel during its first 4 seconds of movement? $v(t) = 3t^2 + 3$
16.	Suppose that *t* minutes after putting 1000 bacteria on a Petri plate the rate of growth of the population is: $r(t) = 7\sqrt[4]{t^3}$ bacteria per minute. (a) Find a formula that describes the number of bacteria populations after the first t minutes? (b) What is the population after 16 minutes?

Solutions

1. $\frac{1}{3}x^3 + K$

2. $-\frac{1}{x} + K$

3. $\frac{3}{4}x^{\frac{4}{3}} + C = \frac{3}{4}\sqrt[3]{x^4} + K$

4. $x^3 + \frac{5x^2}{2} - 7x + K$

5. $12.3x + K$

6. $0.4x^6 - \frac{1}{2}x^2 - 1.25x + K$

7. $\frac{2}{3}\sqrt{x^3} + \frac{1}{8x^2} + K$

8. $6\sqrt[6]{x^{11}} - \frac{4}{5}\sqrt[4]{x} + K$

9. $\frac{1}{3}x^3 - 4x + K$

10. $\frac{1}{5}x^5 - \frac{1}{2}x^2 + K$

11. $-10x^7 + 5x + K$

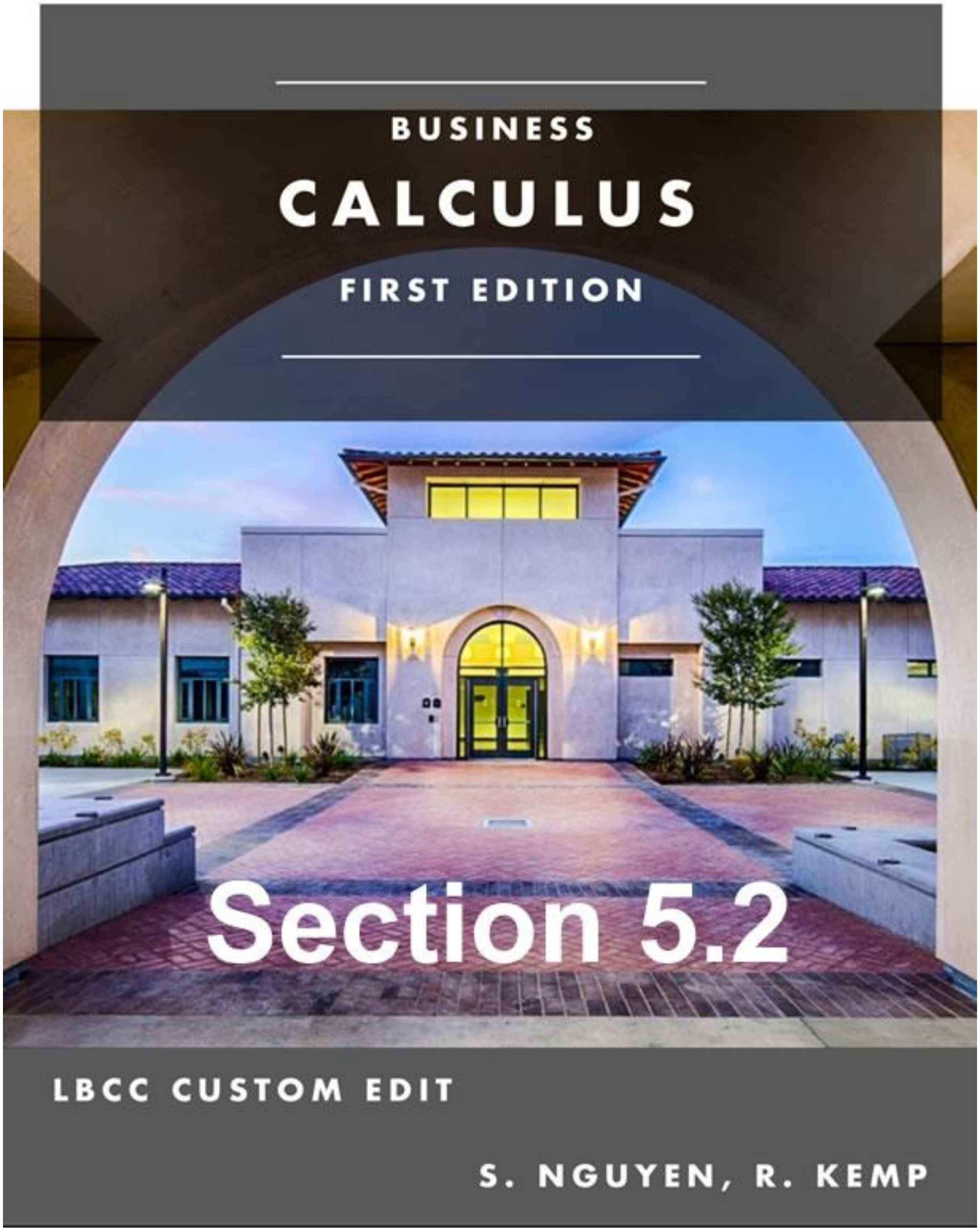
12. $\frac{1}{2}x^2 - 5x + K$

13. $C(x) = 8\sqrt{x} + 2x + 4000$

14. $P(x) = x^4 - x^3 + 6x^{\frac{2}{3}} - 10x - 1200$

15. $D(t) = t^3 + 3t, \quad D(4) = 76\,ft$

16. $4t^{\frac{7}{4}} = 4\sqrt[4]{t^7} + 1000, \qquad 1512\ bacteria$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5.2 - INDEFINITE INTEGRATION WITH EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The Integral of an Exponential Function

Remember that the "Integral' process is an "Anti-Derivative" of a function. Where the **"Derivative" of the "Exponential" function** ($f(x) = e^{ax}$) is given by the following.

$$f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}e^{ax} = ae^{ax}$$

The **"Integral" of the same "Exponential" function** ($f(x) = e^{ax}$), is given by the following result.

$$F(x) = \int f(x)\,dx$$

Exponential Function Integration – with Constant (a)

$$F(x) = \int e^{ax}\,dx = \frac{1}{a}e^{ax} + K$$

Where: ($a \neq 0$)

Exponential Function Integration – with Constant (a = 1)

$$F(x) = \int e^{x}\,dx = e^{x} + K$$

Therefore, the integral of an **"Exponential" function** (**e^{ax}**), where the "exponent" is equal to a constant (a) multiplied by the variable (x), the integration result is equal to the original **"Exponential" function** (**e^{ax}**), divided by the constant (a).

The proof of the above "Integration" of an "Exponential Function", formula is obtained by differentiating to obtain the original an "Exponential Function.

$$f(x) = \frac{F(x)}{dx} = \frac{d}{dx}\left(\frac{1}{a}e^{ax} + K\right) = \frac{1}{a}ae^{ax} = e^{ax}$$

The result of the above differentiation is the original integrand; therefore, the "Integration" of an "Exponential Function" formula is correct. this rule only works if the "exponent" of the function is only a product between a constant and x.

Example Problem #1

1. Find the antiderivative of: $y = e^{7x}$

Solution:

$$y(x) = \int (e^{7x})\, dx = \frac{1}{7} \cdot e^{7x} + K$$

2. Find the antiderivative of: $y = e^{\frac{2}{5}x}$

Solution:

$$y(x) = \int \left(e^{\frac{2}{5}x}\right) dx = \frac{5}{2} \cdot e^{\frac{2}{5}x} + K$$

3. Find the antiderivative of: $y = e^{-3x}$

Solution:

$$y(x) = \int (e^{-3x})\, dx = -\frac{1}{3} \cdot e^{-3x} + K$$

4. Find the antiderivative of: $y = e^{-\frac{7}{8}x}$

Solution:

$$y(x) = \int \left(e^{-\frac{7}{8}x}\right) dx = -\frac{8}{7} \cdot e^{-\frac{7}{8}x} + K$$

Example Problem #2

An investment is growing at a rate of ($A'(t) = 15e^{0.06t}$) dollars per year.

a. Find the formula ($A(t)$) for the total amount of money accumulated in t years, knowing that the Principal amount invested was \$500.

b. Use your formula to find the amount in the account after 10 years.

Solution:

a. $$A(t) = \int A'(t)\,dt = \int 15e^{0.06t}\,dt$$

$$A(t) = 15\left[\frac{e^{0.06t}}{0.06}\right] + K = 250e^{0.06t} + K$$

Next, evaluating ($A(t)$) at ($t = 0\ years$) we obtain the following:

$$A(0) = 250e^{0.06(0)} + K = 250 + K = 500$$

$$K = 500 - 250 = 250$$

Next, replacing K,

$$A(t) = 250e^{0.06t} + 250$$

b. Next, evaluating ($A(t)$) at ($t = 10\ years$)

$$A(10) = 250e^{0.06(10)} + 250 = \$\,705.53$$

Example Problem #3

As of the year 2016, the amount of oil in the world was 1,650 billion barrels and it was predicted to be decreasing at a rate of ($S'(t) = -260.7e^{-0.158t}$) billion barrels per year, where (t) is the number of years since 2016.

Find a formula for the total amount of oil supply in the world t years after 2016.

When will the world be left with 1 billion barrels of oil?

Resource: https://www.worldometers.info/oil/

Solution:

Find the total supply of oil by integrating the rate ($S'(t) = -260.7e^{-0.158t}$):

$$S(t) = \int S'(t)\,dt = \int -260.7e^{-0.158t}\,dt$$

$$S(t) = -260.7\left[\frac{e^{-0.158t}}{-0.158}\right] + K = 1650e^{-0.158t} + K$$

The amount of oil in 2016 was 1,650 billion barrels, so

$$S(0) = 1650e^{-0.158(0)} + K = 1650$$

$$1650 + K = 1650$$

$$K = 0$$

Next, replacing the "Integral Function" ($S(t)$) with the newly derived results yield, the total supply (t) years after 2016:

$$S(t) = 1650e^{-0.158t}$$

Next, to predict when the total world reserves oil will be at 1 billion barrels, we set the above function equal to ($S(t) = 1$) and solve for (t).

$$S(t) = 1650e^{-0.158t} = 1$$

Example Problem #3 – Cont'd

$$e^{-0.158t} = \frac{1}{1650}$$

$$\ln\left(e^{-0.158t}\right) = \ln(.000606)$$

$$-0.158(t) = -7.4085$$

$$t = 46.89\ years \approx 47\ years$$

$$Year = 2016 + 47\ years = 2063$$

Thus, the known world supply of oil will reach 1 billion barrels by year = 2063, soon after the supply will be nearly gone!

Exception to the Power Rule - the Integral ($\int \frac{1}{x}\, dx$)

As we already discussed in the previous section, when the exponent of the function ($x^n = x^{-1}$) equals negative one ($n = -1$), there is an exception to the power rule for integration.

Remember, that the differentiation formula for the natural logarithm of the function of ($f(x) = ln\, x$), is given by the following.

$$\frac{d}{dx} ln\,|x| = \frac{d}{dx} ln|-x| = \frac{1}{x}$$

Therefore, since integration is the reverse of differentiation.

Natural Log (LN) Rule for Integration

$$\int x^n\, dx = \int x^{-1}\, dx = \int \frac{1}{x}\, dx = ln|x| + K$$

Where: ($n = -1$)

Example Problem #4

Find the antiderivative.

$$\int \left(\frac{7}{x}\right) dx = 7\int \frac{1}{x}\, dx = 7\int x^{-1}\, dx = 7ln|x| + K$$

Find the antiderivative.

$$\int \left(\frac{3}{2x}\right) dx = \frac{3}{2}\int \frac{1}{x}\, dx = \frac{3}{2}\int x^{-1}\, dx = \frac{3}{2} ln|x| + K$$

Find the antiderivative.

$$\int \left(\frac{-5}{x}\right) dx = -5\int x^{-1}\, dx = -5ln|x| + K$$

Example Problem #5

Find the antiderivative.

$$g(x) = \int (e^x + 12 - 16x^{-1})dx$$

$$g(x) = \int (e^x)dx + \int 12\,dx - \int 16x^{-1}\,dx$$

$$g(x) = e^x + 12x - 16\,ln|x| + K$$

Example Problem #6

Find the antiderivative.

$$g(x) = \int (8e^{-7x} + 4x^{-1})dx$$

$$g(x) = 8\int (e^{-7x})dx + 4\int x^{-1}\,dx$$

$$g(x) = 8\left(\frac{e^{-7x}}{-7}\right) + 4\,ln|x| + K$$

$$g(x) = -\frac{8}{7}e^{-7x} + 4\,ln|x| + K$$

Example Problem #7

A company is running an ad in the newspaper of a certain city. The company estimates that the amount of people it reaches every is increasing by $\left(P'(t) = \frac{15}{t} + 10t\right)$ people per day.

Let $(t = 1)$ corresponds to the beginning of the ad when no one saw the ad yet.

Find a formula for the total number of people who saw the ad after t days.

Solution:

$$P(t) = \int P'(t)\,dt = \int \left(\frac{15}{t} + 10t\right) dt$$

$$P(t) = 15\ln(t) + 5t^2 + K$$

Next, we evaluate the constant (K) at the given starting time $(t = 1)$; $(P(1) = 0)$:

$$P(1) = 15\ln(1) + 5(1) + K = 0$$

$$P(1) = 0 + 5 + K = 0$$

$$K = -5$$

So,

$$P(t) = 15\ln(t) + 5t^2 - 5$$

Summary

In this section, we have discussed various techniques of Indefinite Integration.

Rules for Indefinite Integration

Constant Multiple Integration Rule - $\int k \cdot f(x)\, dx = k \cdot \int f(x) dx$

Integration of a Constant Rule - $\int k\, dx = kx + K$

Integration Sum - & Difference Rule $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

Integration Power Rule -

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + K$$

Where: ($n \neq -1$)

Natural Log (LN) Rule for Integration

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = ln|x| + K$$

Exponential Function Integration – with Constant (a)

$$C(x) = \int e^{ax}\, dx = \frac{1}{a} e^{ax} + K$$

Where we defined the process of integration as being the reverse of the process of differentiation; and allows you to recover any quantity from its rate of change.

The business case for integration allows you to recover the profit, revenue, and cost functions, given the marginal profit, revenue, and cost functions.

5.2 - EXERCISES

Find each indefinite Integral			
1.	$\int e^{0.06x}\,dx$	2.	$\int 500e^{\frac{x}{5}} + 100e^{-0.05x}\,dx$
3.	$\int x^{-2} - x^{-1}\,dx$	4.	$\int (\frac{3}{x} + x^3 - 6x)dx$
5.	$\int \left(\sqrt{v} + e^v - \frac{1}{4v}\right)dv$	6.	$\int (-3e^{-x} - 6x^{-1})dx$
7.	$\int \left(e^{0.5x} + \frac{10}{x}\right)dx$	8.	$\int \frac{(2t+3)(3t-1)}{4t^2}dt$
9.	$\int (3x^2 + 2x + 1 + x^{-1} - x^{-2})dx$	10.	$\int \left(\frac{x^2 - 6x + 8}{x}\right)dx$
11.	The value of a car is depreciating at a rate of $\mathrm{P}'(\mathrm{t})$. $\mathrm{P}'(\mathrm{t}) = -3{,}240e^{-0.09t}$ Knowing that the purchase price of the car was \$36,000, find a formula for the value of the car after t years. Use this formula to find the value of the car 10 years after it has been purchased.		

12.	Find the price–demand equation (i.e. the equation of price as a function of demand) for a particular brand of toothpaste at a supermarket chain, knowing that the demand is 50 tubes per week at a price of $2.35 per tube, given that the **marginal** price—demand function, $p'(x)$, for x number of tubes per week, is given as $$p'(x) = -0.015e^{-0.01x}$$ If the supermarket chain sells 100 tubes per week, what price should it set?
13.	Suppose a population of fruit flies increases at a rate of $g(t)$, in flies per day. $$g(t) = 2e^{0.02t}$$ If the initial population of fruit flies is 100 flies, find the equation of the total fruit flies' population after t days?

Solutions

17. $\frac{50}{3}e^{0.06x} + K$

18. $2500e^{\frac{x}{5}} - 2000e^{-0.05x} + K$

19. $-x^{-1} - ln|x| + K$

20. $3ln|x| + \frac{x^4}{4} - 3x^2 + K$

21. $\frac{2}{3}v^{\frac{3}{2}} + e^v - \frac{1}{4}ln|v| + K$

22. $3e^{-x} - 6ln|x| + K$

23. $2e^{0.5x} + 10ln|x| + K$

24. $\frac{3}{2}t + \frac{7}{4}ln|t| + \frac{3}{4t} + K$

25. $x^3 + x^2 + x + ln|x| + \frac{1}{x} + K$

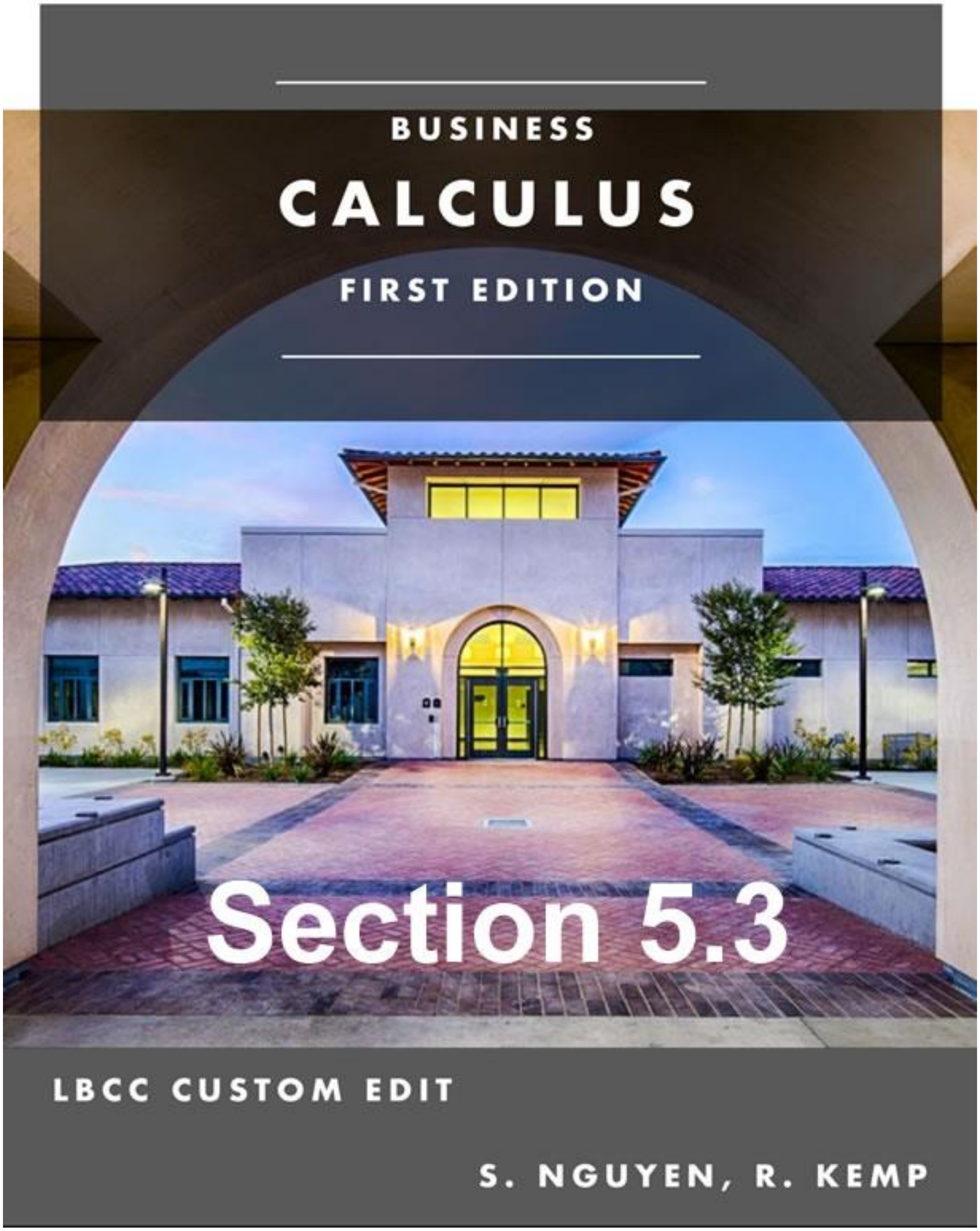
26. $\frac{x^2}{2} - 6x + 8ln|x| + K$

27. $P(t) = 36{,}000e^{-0.09t}$, \$14, 636.51

28. $p(x) = 1.5e^{-0.01x} + 1.44,$

$The\ price\ should\ be\ \$1.99\ for\ 100\ tubes\ per\ week.$

29. $G(t) = 100e^{0.02t}$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5.3 - THE RIEMANN SUM - DEFINITE INTEGRATION

Introduction

This section deals with the concepts of bound areas beneath curves, and starts with the simple geometric idea of area, and applies the techniques of **Definite Integral Calculus**, also known as the **Riemann Summation - Theorem of Calculus**.

Similar, to the concepts applied to "Indefinite Integral Calculus", the concepts of "**Riemann Sum - Definite Integral Calculus**" will be developed into a combination of theory, techniques, and applications.

The work of **Definite Integration** calculus, is the reverse process of taking a derivative, also known as antidifferentiation; which, for a given derivative, restores the original function. The result of the **definite integral**, is an "area" bound between the limits of integration ($\int_a^b dx$) along the domain (x).

We use the symbol ($\int_a^b dx$) to denote the "Limits" an integration along a fixed interval ($a \rightarrow b$) of a horizontal axis or domain. Later we will use the limits of integration for other purposes, such as finding "areas" beneath curves.

The great German mathematician Georg Bernhard Riemann (1826 – 1866); predicted the limit of the "Riemann Sum", is the number of (n) rectangles approaching infinity, gives the area under the curve, and is called the Definite Integral of the function ($f(x_n)$), which is evaluated from the limits of the fixed interval ($a \rightarrow b$), and along the horizontal axis or domain; and is expressed mathematically:

Riemann sum – Definite Integral

$$A(x) = \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

A "**Riemann sum"** for a function ($f(x)$) over an interval [***a, b***] is a sum of areas of rectangles that approximates the area under the curve.

A Definite Integration Describes – The Area Beneath a Curve

What does the process of "Definite Integration" predict? The process of "definite integration" predicts the "area" beneath a curve.

If you look on the inside cover of nearly any traditional math book, you will find various formula, for the geometric quantity of known as "Area" (A), such as:

- Area of a square/rectangle – $A = L \times W$ $\rightarrow$ ft^2,
- Area of a triangle – $A = \frac{1}{2}b \times h$ $\rightarrow$ m^2,
- Area of a circle – $A = \pi r^2$ $\rightarrow$ in^2.

The unit of measure for the quantity area ($A \rightarrow$ ft^2, m^2, in^2) is distance squared, or squared units.

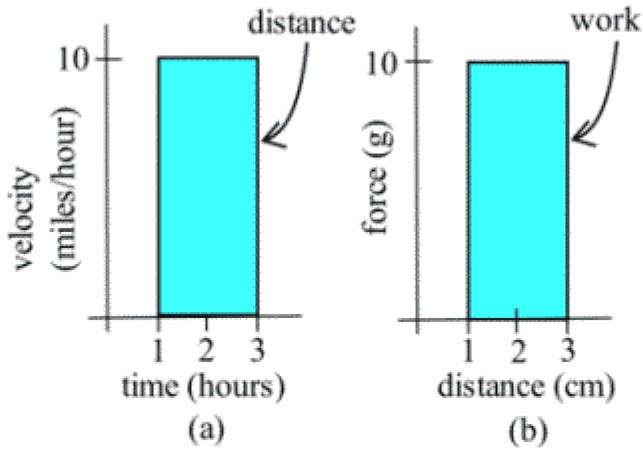
For example, when considering the "area of a rectangle" ($A = L \times W$); each side of the rectangle is a distance measurement; where the units of measure could be: *meters, feet, inches, centimeters*, and so on.

The area of the rectangle ($A = L \times W$) is calculated by multiplying together each side of the rectangle, which results in square "area units":

- *meters* × *meters* = *square meters* = m^2
- *feet* × *feet* = *square feet* = ft^2
- *inches* × *inches* = *square inches* = in^2,

One reason "areas" are so useful for "integration calculus" is that they can represent quantities other than simple geometric shapes.

Mathematically, the "rectangular area" is the basic shape, which provides a useful tool for performing an integration or antidifferentiation; because the "area" of a rectangle is simply the product of the base×height – ($A = b \times h$) of the rectangle.



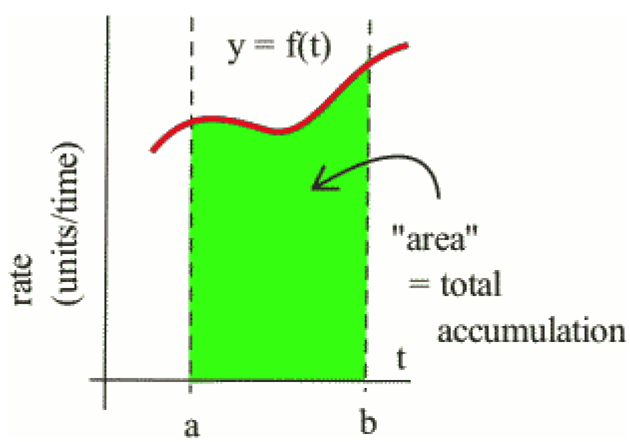
Considering the left-side "rectangular area" image above, if the **"base"** of the rectangle is a measure of "time" whose units are *hours*, and the "height" of the rectangle is the measure of "velocity" whose units are *miles/hour*, then the "area units" of the rectangle are ***hours*×*miles/hour = miles***, a measure of distance.

Velocity (miles/hour) × Time (hour) = Distance (miles)

Similarly, on the right-side "rectangular area" image above, if the **"base"** of the rectangle is a measure of a "distance" whose units are *centimeters (cm),* and the **"height"** of the rectangle is a measure of "force" whose units are in *grams (g)*, then the "area units" are ***gram-centimeters***, a measure of work.

Force (g) × Distance (cm) = Work (g × cm)

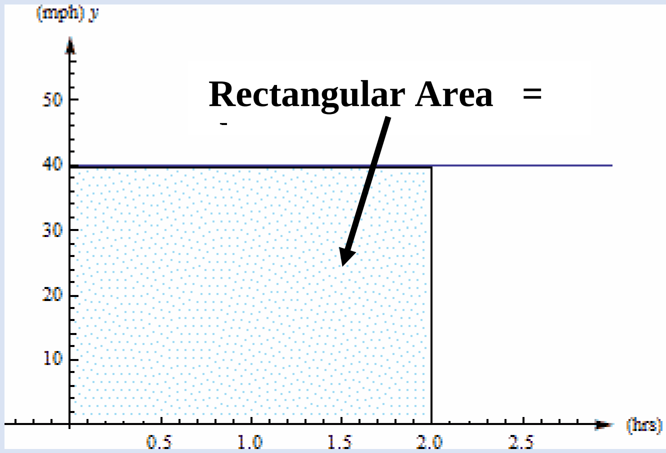
The mathematical process of **"Integration"** measures the total amount of accumulation of "area" beneath the curve (***y = f(t)***). For functions representing other rates such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the total amount of something.



Next, we will demonstrate in the examples below, functions representing other rates which predict "Areas" which could be found beneath a curve.

Example Problem #1

Suppose a car travels on a straight road, at a constant rate of Velocity (v = 40 miles per hour) for a Time of two hours (t = 2 hours). See the graph of its velocity vs time below. How far in distance has the car traveled?



$$\text{Rectangular Area} = \text{height} \times \text{base}$$

$$\mathbf{d = v \times t}$$

$$\mathbf{d = 40\ miles/hour \times 2\ hour = 80\ miles}$$

In the example, the function is represented by the "velocity rate (v)" of motion, and the "distance" ($\mathbf{d = v \times t}$), a car travels moving at a constant velocity rate, equals to the "velocity rate (v)" multiplied by the "time (t)", and the "area" is represented the total distance (d) traveled by the car.

The mathematics predict, the car has traveled (80 miles).

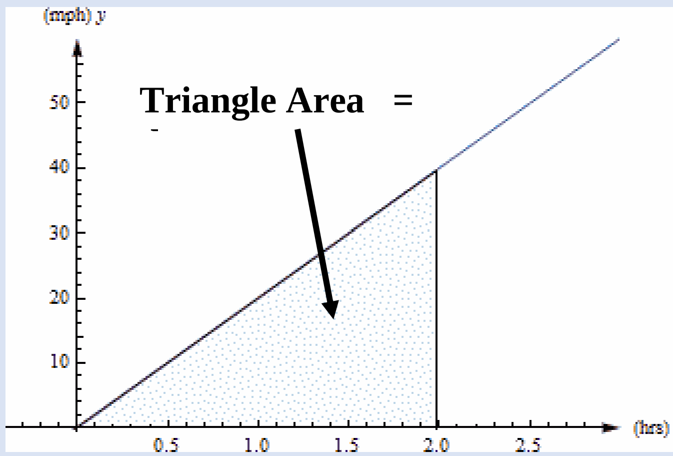
The above problem demonstrates that using the concept of "Rates of Motion", a "**Rectangular Area**", could be defined; and used to describe the area beneath a curve.

The trouble with relying only on "rectangular areas" and the simple geometry of the product of the base×height – ($A = b \times h$) of the rectangle, is that the equation only works if the rate is constant.

However, if the rate is changing, or the area is the shape of a triangle, or some other shape, then the above equation will not accurately predict the accumulation of area beneath a curve. The next example will demonstrate a "Triangular Area".

Example Problem #2

Now suppose that a car travels so that its speed increases steadily from "velocity" (v = 0 to v = 40 miles per hour), for a "time" of two hours (t = 2 hours). See the graph of its velocity in below. How far has this car traveled?



The graph describes a **"triangular area"** between the **"height axis"** measured by the **"velocity rate (v)"**, vs the **"base axis"** which is a **"time interval (t)"**, between (t = 0 and t = 2 hours); and the **distance (d)** is the triangular **area** beneath a curve.

$$\text{Area} = \frac{1}{2}(height \times base)$$

The distance (d) the car travels is described by the "triangular area" between its "velocity rate (v)" vs the **"time interval (t)"**, between (t=0 and t=2 hours)

$$\text{d} = \frac{1}{2}(\text{v} \times t)$$

$$\text{d} = \frac{1}{2}(40\, miles/hour \times 2\, hour) = 40 \text{ miles}$$

The car travels a total distance of (d = 40 miles) during its trip.

In the example, the function is represented a "velocity rate (v)" of travel (miles per hour), and the area represented the total distance traveled.

Riemann Sum - Definite Integral

The Definite Integral of the function ($f(x_n)$), which is evaluated from the limits of the fixed interval ($a \to b$), and along the horizontal axis or domain; and is expressed mathematically:

Riemann sum – Definite Integral

$$A(x) = \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

A "**Riemann sum"** for a function ($f(x)$) over an interval [***a, b***] is a sum of areas of rectangles that approximates the area under the curve.

Start by dividing the interval [***a, b***] into (n) subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width (Δx).

The height of each rectangle comes from the function ($f(x_n)$), evaluated at some point (x_n) in its sub interval.

Then the Riemann sum is the approximating area ($A(x)$) under a non-negative function ($f(x_n)$), by (n) rectangles means multiplying heights ($f(x_n)$) by widths (Δx) and adding each function; as in the following.

$$A(x) = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots\cdots + f(x_n) \cdot \Delta x$$

or, factoring out the (Δx),

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = [f(x_1) + f(x_2) + f(x_3) + \cdots\cdots + f(x_n)] \cdot \Delta x$$

The upper-case Greek letter Sigma ($\sum$) is used to stand for sum. Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.

The **Riemann sum** is the approximating area ($A(x)$) under a generalized curve, of function ($f(x)$).

Area Beneath Function (*f*) from [a, b] Approximated by a number of (n) Rectangles

1. **Rectangular Width**

$$\Delta x = \frac{b - a}{n}$$

2. **Location Along Domain and Interval [*a, b*]**

$$[x_i = a + i \cdot \Delta x] \quad \text{or} \quad [x_{i+1} = x_i + \Delta x]$$

$$[x_0 = a] \quad ; \quad [x_n = b]$$

3. **Riemann Sum (Area)**

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x$$

4. **Definite Integral**

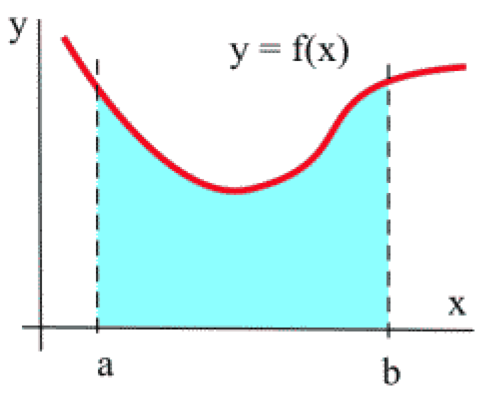
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} [f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x]$$

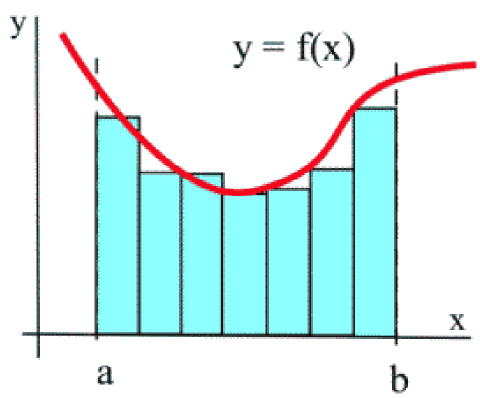
Riemann sum – Definite Integral

$$A(x) = \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

Approximating An "Integration Area" By Rectangles

The diagram below shows a non-negative function (f(x)). We want to calculate the area under the curve and above the x-axis domain, and between the vertical lines where, the numbers (***a***) and (**b**) are called the **lower limit** and **upper limits of integration**, respectively.

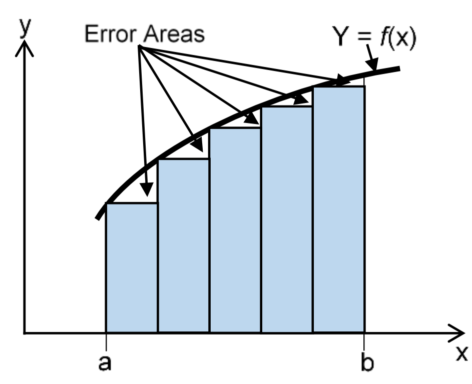




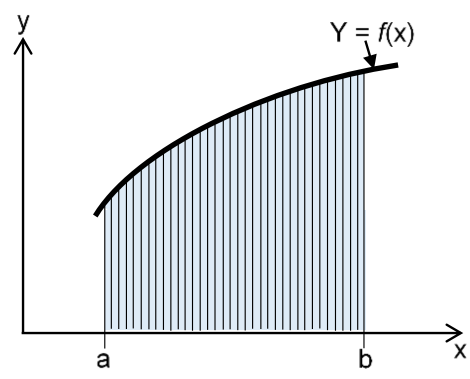
To approximate the "**area**" if the rate curve is, curvy, we use rectangles and triangles, like we did in the last example.

But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.

Suppose we want to calculate the area between the graph of a positive function ($y = f(x)$) and the (x)-axis on the interval [***a, b***] (graphed below).



Area beneath curve and function (y = f(x)) from interval [a to b] approximated by five (5) rectangles



Area beneath curve and function (y = f(x)) from interval [a to b] approximated by thirty-five (35) rectangles

The Riemann Sum method is to build several rectangles with bases on the interval [***a, b***] and left/right sides that reach up to the graph of (f) (see above).

Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of ($f(x)$) on [***a, b***].

The area of the region formed by the rectangles approximates the area we want.

We begin by approximating the area by rectangles.

In the graph on the left above, the area under the curve is approximated by “five” rectangles with equal bases, and with heights equal to the height of the curve at the left-hand edge of the rectangle. (These are called left rectangles.)

Five rectangles, however, do not give a very accurate approximation for the area under the curve: They underestimate the actual area by the small white spaces just above the rectangles.

In the second graph on the right (above), this same area is approximated by “thirty-five (35)” rectangles, giving a much better approximation: the white “error area” between the curve and the rectangles is so small as to be almost invisible.

These diagrams suggest that more rectangles give a better approximation.

In fact, the exact area under the curve is defined as the limit of the approximations as the number of rectangles approach infinity.

The following Example shows how to carry out such an approximation for the area under the curve, and afterward we will find the exact area by letting the number of rectangles approach infinity.

Approximating the "Integration Area" Using Left & Right Side Rectangles

In the graphs, the "**left**" sides of the rectangles $(y = f(x_{i-1}))$; are called "**left side rectangles**"; and are approximated differently from the "**right**" sides of the rectangle, these are called "**right side rectangles**" $(y = f(x_i))$.

.

1. **Location Along Domain and Interval [*a, b*]**

Right Hand Edges **Left Hand Edges**

$$[x_i = a + i \cdot \Delta x] \quad \text{or} \quad [x_{i-1} = a + (i-1) \cdot \Delta x]$$

$$[x_0 = a] \quad ; \quad [x_n = b]$$

2. **Riemann Sum (Area) – Left Hand Edges**

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

$$\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_{n-1}) \cdot \Delta x$$

3. **Riemann Sum (Area) – Right Hand Edges**

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots\cdots + f(x_n) \cdot \Delta x$$

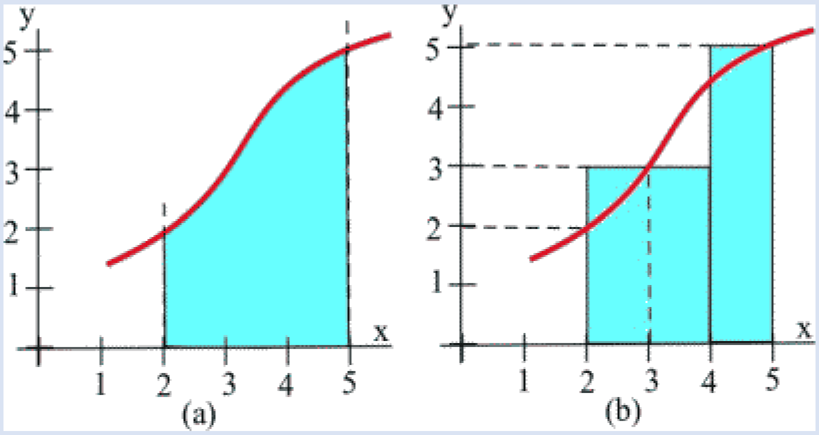
4. **Riemann Sum (Area) – Right Hand Edges**

$$A = R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$A = L_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

Example Problem #3

Approximate the area in the graph below, on the left between the graph of (f) and the x-axis, on the interval [2, 5], by summing the areas of the rectangles in the graph on the right.



You can find the “area” of each rectangle using the (**Area = Height x Width**) equation.

The total area of rectangles is:

$$Area = (1)\cdot(3) + (1)\cdot(3) + (1)\cdot(5) = 11\, units^2$$

Or

$$Area = (2)\cdot(3) + (1)\cdot(5) = 11\, units^2$$

Example Problem #4

Approximate the area under the curve ($f(x) = x^2$), on the interval from [1 to 2] by five (5) rectangles.

Use rectangles with equal bases and with heights equal to the height of the curve at the ***"Left-Hand"*** and ***"Right-Hand"*** edge of the rectangles.

Solution:

c. Find the Rectangle Width. For the five rectangles ($n = 5$), we divide the distance from [***a*** = 1 to ***b*** = 2] into five equal parts, so each rectangle has a change in x, width – (Δx).

$$\Delta x = \frac{b - a}{n} = \frac{2 - 1}{5} = 0.2$$

d. Using ***"Right-Hand"*** edges, find the distance from [***a*** = 1 to ***b*** = 2], for each of the five rectangles ($n = 5$).

$$[x_i = \boldsymbol{a} + i \cdot \Delta x] \quad ; \quad [x_n = \boldsymbol{b}]$$

$$[x_i = 1 + i \cdot (0.2)] \quad ; \quad [x_5 = \boldsymbol{b} = 2]$$

$$\begin{aligned}
x_1 &= \boldsymbol{a} + \Delta x &&= 1.0 + 0.2 &&= 1.2 \\
x_2 &= \boldsymbol{a} + 2 \cdot \Delta x &&= 1.0 + 2 \cdot (0.2) &&= 1.4 \\
x_3 &= \boldsymbol{a} + 3 \cdot \Delta x &&= 1.0 + 3 \cdot (0.2) &&= 1.6 \\
x_4 &= \boldsymbol{a} + 4 \cdot \Delta x &&= 1.0 + 4 \cdot (0.2) &&= 1.8 \\
x_5 &= \boldsymbol{b} = \boldsymbol{a} + 5 \cdot \Delta x &&= 1.0 + 5 \cdot (0.2) &&= 2.0
\end{aligned}$$

---

e. Using ***"Left-Hand"*** edges, find the distance from [***a*** = 1 to ***b*** = 2], for each of the five rectangles ($n = 5$).

$$[x_{i-1} = \boldsymbol{a} + (i-1) \cdot \Delta x] \quad ; \quad [x_{n-1} = \boldsymbol{b - 1}]$$

$$[x_{i-1} = 1 + (i-1) \cdot (0.2)] \quad ; \quad [x_4 = \boldsymbol{2 - 1} = 1]$$

$$\begin{aligned}
x_0 &= \boldsymbol{a} + 0 \cdot \Delta x &&= 1.0 + 0 &&= 1 \\
x_1 &= \boldsymbol{a} + 1 \cdot \Delta x &&= 1.0 + 1 \cdot (0.2) &&= 1.2 \\
x_2 &= \boldsymbol{a} + 2 \cdot \Delta x &&= 1.0 + 2 \cdot (0.2) &&= 1.4 \\
x_3 &= \boldsymbol{a} + 3 \cdot \Delta x &&= 1.0 + 3 \cdot (0.2) &&= 1.6 \\
x_4 &= \boldsymbol{a} + 4 \cdot \Delta x &&= 1.0 + 4 \cdot (0.2) &&= 1.8
\end{aligned}$$

Example Problem #4 – Cont'd

Solution:

f. Find the Area beneath the curve. The Area beneath the curve is equal to the sum of the five rectangles using the ***"Left-Hand"*** edge areas.

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

$$\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_{n-1}) \cdot \Delta x$$

$$\sum_{i=1}^{5} x_{i-1}^2 \cdot \Delta x = x_0^2 \cdot \Delta x + x_1^2 \cdot \Delta x + x_2^2 \cdot \Delta x + x_3^2 \cdot \Delta x + x_4^2 \cdot \Delta x$$

$$\sum_{i=1}^{5} x_{i-1}^2 \cdot \Delta x = (1)^2 \cdot (0.2) + (1.2)^2 \cdot (0.2) + (1.4)^2 \cdot (0.2) + (1.6)^2 \cdot (0.2) + (1.8)^2 \cdot (0.2)$$

$$\sum_{i=1}^{5} x_{i-1}^2 \cdot \Delta x = 0.2 + 0.288 + 0.392 + 0.512 + 0.648 = 2.04$$

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = 2.04$$

Therefore, the sum of the areas of the five rectangles, is equal to the area beneath the curve; and is an "**Left Side Rectangle**" area equal to the height times base; approximately equal to **2.04 square units**.

Example Problem #4 – Cont'd

Solution:

g. Find the Area beneath the curve. The Area beneath the curve is equal to the sum of the five rectangles; using the ***"Right-Hand"*** areas.

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x$$

$$\sum_{i=1}^{5} x_i^2 \cdot \Delta x = x_1^2 \cdot \Delta x + x_2^2 \cdot \Delta x + x_3^2 \cdot \Delta x + x_4^2 \cdot \Delta x + x_5^2 \cdot \Delta x$$

$$\sum_{i=1}^{5} x_i^2 \cdot \Delta x = (1.2)^2 \cdot (0.2) + (1.4)^2 \cdot (0.2) + (1.6)^2 \cdot (0.2) + (1.8)^2 \cdot (0.2) + (2)^2 \cdot (0.2)$$

$$\sum_{i=1}^{5} x_i^2 \cdot \Delta x = 0.288 + 0.392 + 0.512 + 0.684 + 0.80 = 2.64$$

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = 2.64$$

Therefore, the sum of the areas of the five rectangles, is equal to the area beneath the curve; and is an "**Right Side Rectangle**" area equal to the height times base; approximately equal to **2.64 square units**.

The true area is the average value, between these two estimates.

So, we could take their average:

$$Average = \frac{L_n + R_n}{2} = \frac{2.04 + 2.64}{2} = \frac{4.68}{2} = 2.34$$

In general, the average of the "**Left-Hand**" and "**Right-Hand**" estimates will be closer to the real area than either individual estimate.

Our estimate of the area under the curve is about 2.34 square units.

$$Average = \frac{L_n + R_n}{2} = 2.34 \ square\ units$$

(The actual area is about 2.3333.)

As demonstrated, using only five rectangles does not give a very accurate approximation for the true area beneath the curve.

For greater accuracy if we use more rectangles, the area beneath the curve fits the curve in an exacting way.

The table below demonstrated the "rectangular approximation" for the area beneath the curve, in the above example. Generating a larger number of rectangles, with answers rounded to three decimal places.

The areas in the column are approaching 2.33333, which is the exact area under the curve.

Therefore, the area under the curve ($f(x) = x^2$), a distance [***a*** = 1 to ***b*** = 2] is equal to 2.333 square units.

As we saw earlier, using "five" rectangles do not give a very accurate approximation for the true area under the curve.

For greater accuracy, we use more rectangles, calculating the area in the same way.

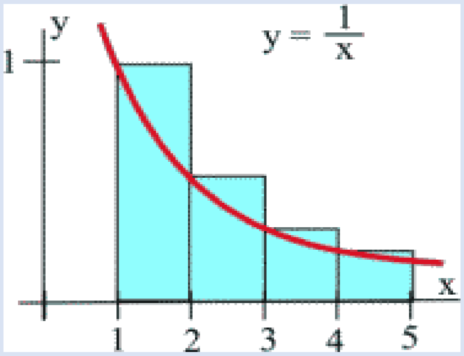
The following table gives the "rectangular approximation" for the area under the curve in Example 1 for larger numbers of rectangles with answers rounded to three decimal places.

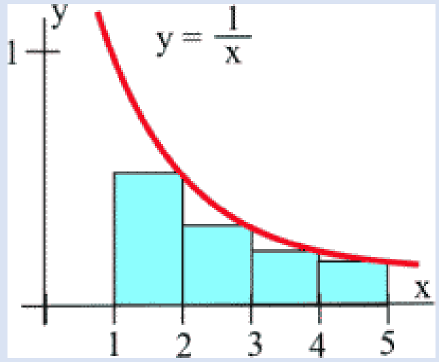
The areas in the right-hand column are approaching 2.333, which is the exact area under the curve. Therefore, the area under the curve (f(x) = x^2) from (1 to 2) is 2.5 square units.

Areas are given in "square units", meaning that if the units on the graph are inches, feet, or some other units, then the area is in "square inches", "square feet", or some other general square units.

Example Problem #5

Below are graphs showing two ways, to use four rectangles to approximate an area. The first use the "Left-Hand" edge; the second uses the "Right-Hand" edge.





Let A be the area region, bounded by the graph of the function ($f = \frac{1}{x}$), the x-axis, and vertical lines at [**x**=1 and **x**=5].

When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere.

But it is easiest to choose rectangles that (a) have all the same width and (b) take their heights from the function at one edge. We will use the Riemann Sum to calculate the total area of the "**Left Side rectangles**".

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x = 2.04$$

$$\sum_{i=1}^{4} f(x_{i-1}) \cdot \Delta x = f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x$$

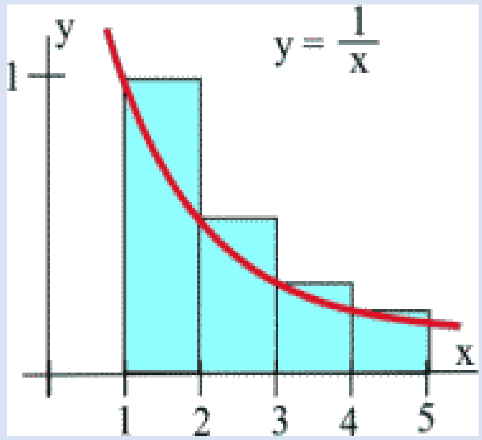
Below are graphs showing two ways, to use four rectangles to approximate this area.

In the first graph, we used left-endpoints; the height of each rectangle comes from the function value at its left edge.

In the second graph on the next page, we used right-hand endpoints.

Example Problem #5 – Cont'd

Left-hand endpoints: The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function ($f = \frac{1}{x}$), and each rectangle has a width of 1.



You can find the "area" of each rectangle using the (**Area = Height x Width**) equation.

The total area of the rectangles, using the "***Left-Hand"*** estimate of the area under the curve, is:

$$L_n = \sum_{i=1}^{4} \frac{1}{x_{i-1}} \cdot \Delta x = \frac{1}{x_0} \cdot \Delta x + \frac{1}{x_1} \cdot \Delta x + \frac{1}{x_2} \cdot \Delta x + \frac{1}{x_3} \cdot \Delta x$$

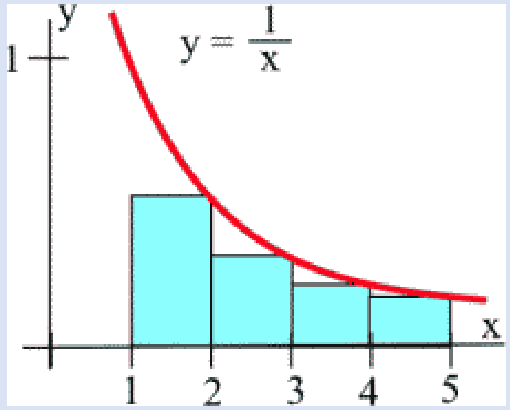
$$\sum_{i=1}^{4} \frac{1}{x_{i-1}} \cdot \Delta x = \left(\frac{1}{1}\right)^2 \cdot (1) + \left(\frac{1}{2}\right)^2 \cdot (1) + \left(\frac{1}{3}\right)^2 \cdot (1) + \left(\frac{1}{4}\right)^2 \cdot (1)$$

$$L_n = \sum_{i=1}^{4} \frac{1}{x_{i-1}} \cdot \Delta x = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{25}{12} \approx 2.08$$

Notice that because this function is decreasing, all the "**left**" endpoint rectangles stick out above the region we want – using "**Left-Hand**" endpoints will overestimate the area.

Example Problem #5 – Cont'd

Right-hand endpoints: The right-hand estimate of the area is



The total area of the rectangles, using the "**right-hand**" estimate of the area under the curve, is given below:

You can find the "total area" of the **"Right-Hand"** side using the Riemann Summation equation.

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$\sum_{i=1}^{4} f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + f(x_4) \cdot \Delta x$$

$$\sum_{i=1}^{4} \frac{1}{x_i} \cdot \Delta x = \frac{1}{x_1} \cdot \Delta x + \frac{1}{x_2} \cdot \Delta x + \frac{1}{x_3} \cdot \Delta x + \frac{1}{x_4} \cdot \Delta x$$

$$\sum_{i=1}^{4} \frac{1}{x_i} \cdot \Delta x = \left(\frac{1}{2}\right)^2 \cdot (1) + \left(\frac{1}{3}\right)^2 \cdot (1) + \left(\frac{1}{4}\right)^2 \cdot (1) + \left(\frac{1}{5}\right)^2 \cdot (1)$$

$$R_n = \sum_{i=1}^{4} \frac{1}{x_i} \cdot \Delta x = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{77}{60} \approx 1.28$$

All the **"Right-hand"** rectangles lie completely under the curve, so this estimate will be an underestimate.

Example Problem #5 – Cont'd

We can see that the true area is the average value, between these two estimates.

So, we could take their average:

$$Average = \frac{L_n + R_n}{2}$$

$$Average = \frac{\frac{25}{12} + \frac{77}{60}}{2} = \frac{2.08 + 1.28}{2} = \frac{101}{60} = \frac{3.36}{2} \approx 1.68$$

In general, the average of the "**Left-Hand**" and "**Right-Hand**" estimates will be closer to the real area than either individual estimate. These sums of areas of rectangles are called Riemann sums.

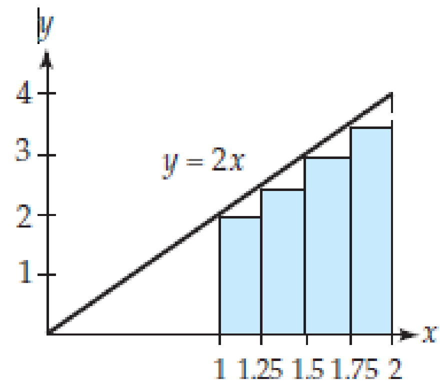
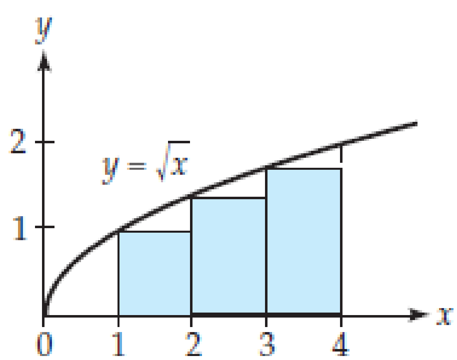
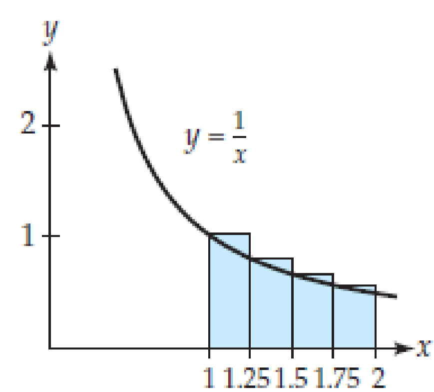
Our estimate of the area under the curve is about 1.68.
(The actual area is about 1.61.)

If we wanted a better answer, we could use even more and even narrower rectangles. But there is a limit to how much work we want to do by hand.

5.3 - EXERCISES

Find the sum of the areas of the shaded rectangles under each graph. Round to two decimal places.

[Hint: The width of each rectangle is the difference between the x-values at its base. The height of each rectangle is the height of the curve at the left edge of the rectangle.]

1.	
2.	
3.	

Approximate the Area under the curve from (***a***) to (***b***) by calculating the Riemann Sum with the given number of rectangles (**n**); rounding to three decimal places.			
4.	$f(x) = 3x$ from ***a*** = 1 to ***b*** = 2 use **Left-Hand** side and 5 rectangles	5.	$f(x) = x^2 + 2$ from ***a*** = 0 to ***b*** = 1 use **Right-Hand** side and 6 rectangles
6.	$f(x) = e^x$ from ***a*** = -1 to ***b*** = 1 use **Average value** and 7 rectangles	7.	$f(x) = \sqrt{x}$ from ***a*** = 1 to ***b*** = 5 use **Left-Hand** side and 5 rectangles
8.	$f(x) = \frac{1}{\sqrt{x}}$ from ***a*** = 1 to ***b*** = 8 use **Right-Hand** side and 7 rectangles	9.	$f(x) = \frac{5}{x}$ from ***a*** = 1 to ***b*** = 2 use **Average value** and 5 rectangles
10.	$f(x) = x^2$ from ***a*** = -2 to ***b*** = 2 use **Left-Hand** side and 4 rectangles	11.	$f(x) = x^3$ from ***a*** = 0 to ***b*** = 2 use **Right-Hand** side and 4 rectangles

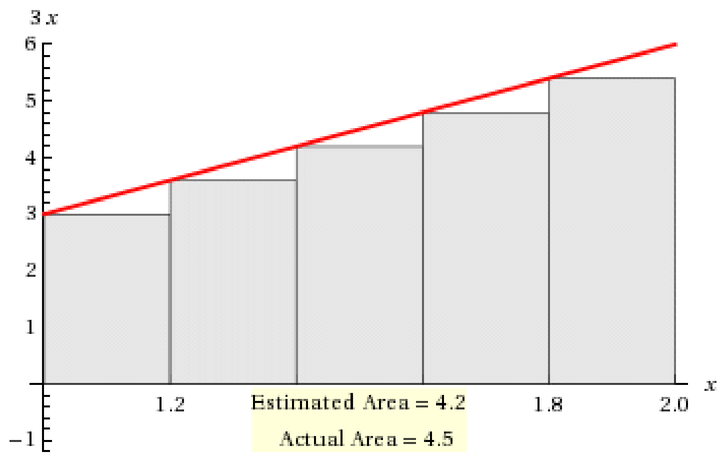
Solutions

30. $2.75\ square\ units$

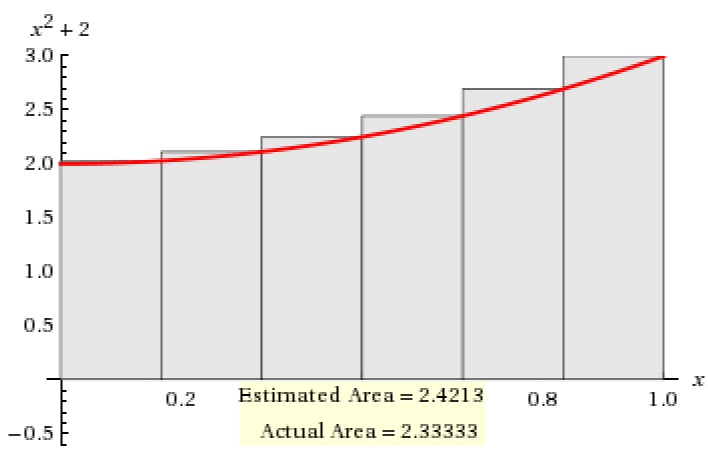
31. $4.15\ square\ units$

32. $0.760\ square\ units$

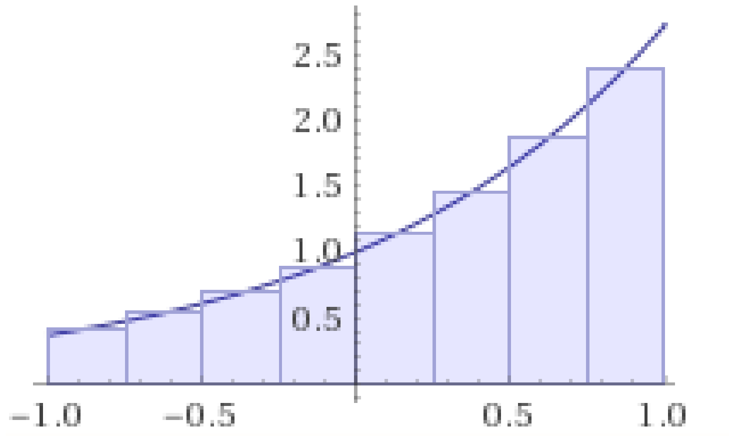
33. $4.20\ square\ units$



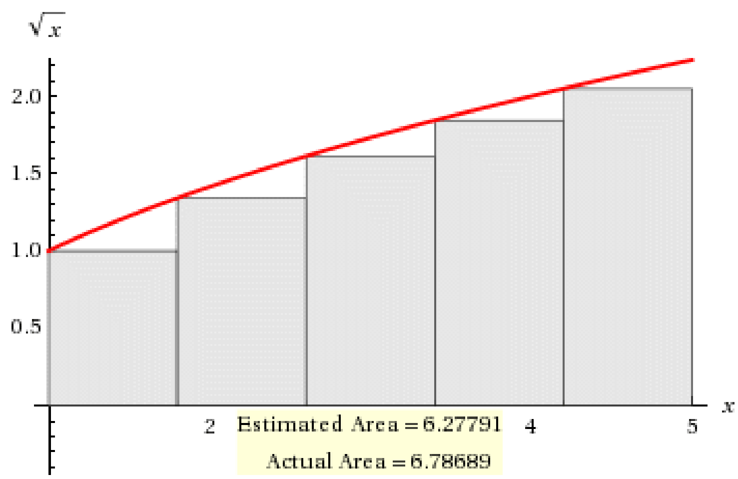
34. $2.42\ square\ units$



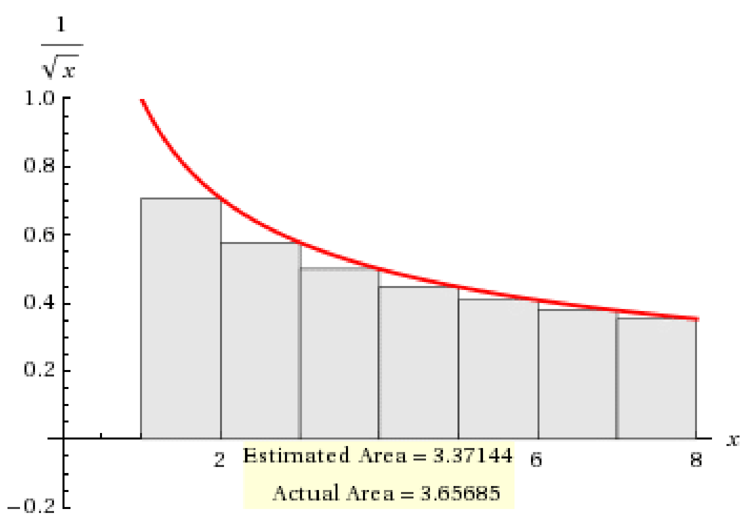
35. 2.36 *square units*



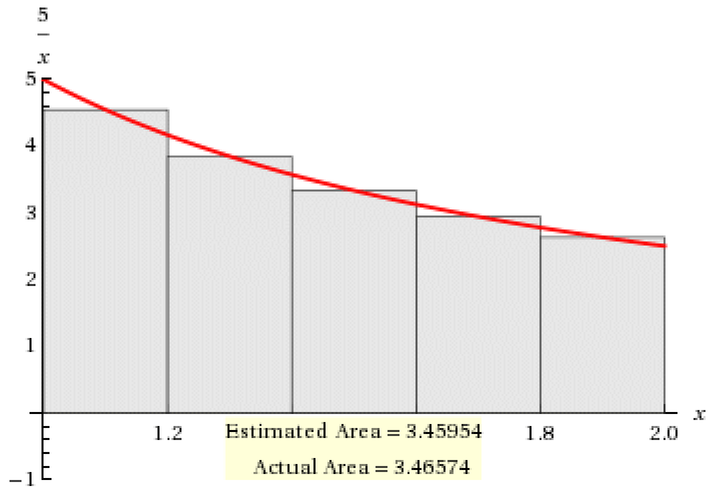
36. 6.28 *square units*



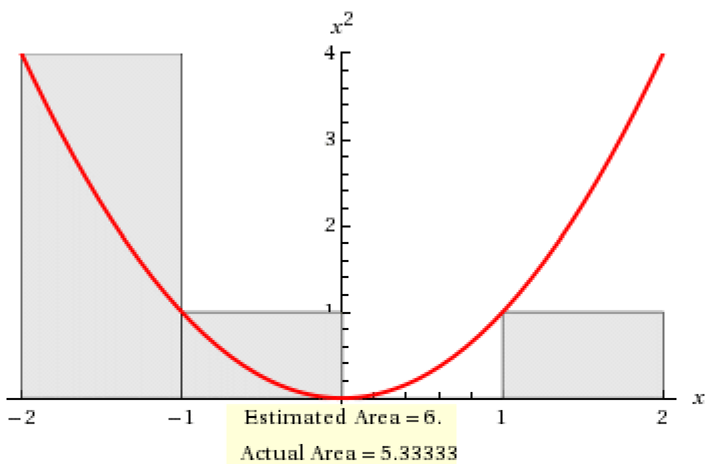
37. 3.37 *square units*



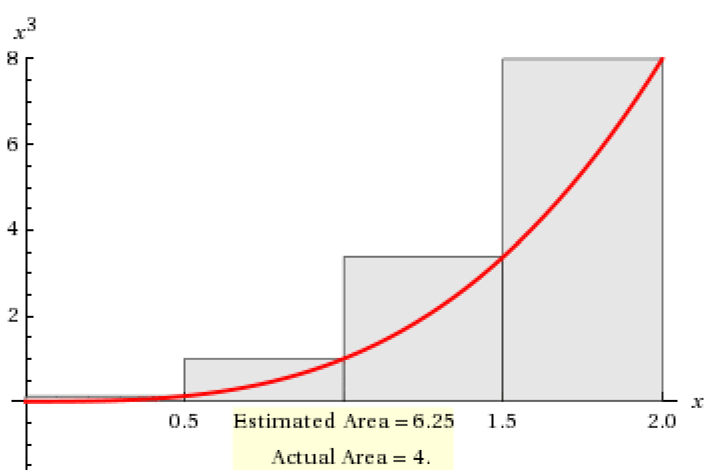
38. 3.47 *square units*

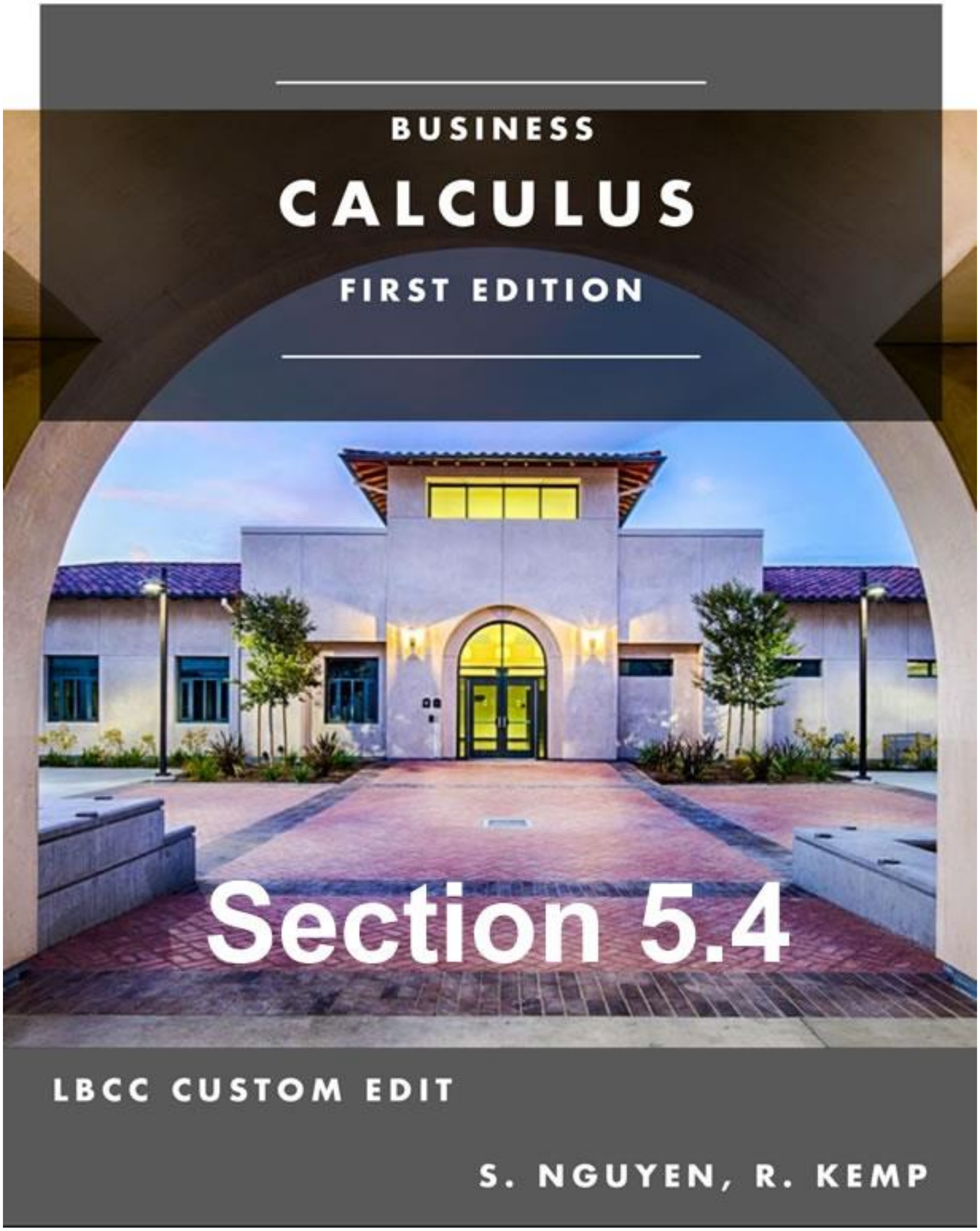


39. 6 *square units*



40. 6.25 *square units*

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5.4 - THE DEFINITE INTEGRATION

Introduction

This section deals with the concepts of bound areas beneath curves and starts with the simple geometric idea of finding an area. We will apply the techniques of **Definite Integral Calculus** and see how this concept ties in with the **Differential Calculus** through the **Fundamental Theorem of Calculus**.

Like the concepts applied to "Indefinite Integral Calculus", the concepts of "**Definite Integral Calculus**" will be developed into a combination of theory, techniques, and applications.

The work of **Definite Integration** calculus, is the reverse process of taking a derivative, also known as antidifferentiation; which, for a given derivative, restores the original function. The result of the **definite integral**, is an "area" bound between the limits of integration **a** and **b** ($\int_a^b dx$) along the domain (x).

We use the symbol ($\int_a^b dx$) to denote the "Limits" an integration along a fixed interval ($a \rightarrow b$) of a horizontal axis or domain. Later we will use the limits of integration for other purposes, such as finding "areas" beneath curves.

For example, the "definite integral" of a marginal cost function ($MC(x) = \frac{dC(x)}{dx}$), turns a marginal cost back into a cost function; which is a bound "area" between the limits of the fixed interval ($a \rightarrow b$), and along the (x) domain.

$$C(x) = \int_a^b MC(x)\, dx$$

This definite integral will compute the accumulated Cost between the input values a and b.

A Definition of Definite Integral

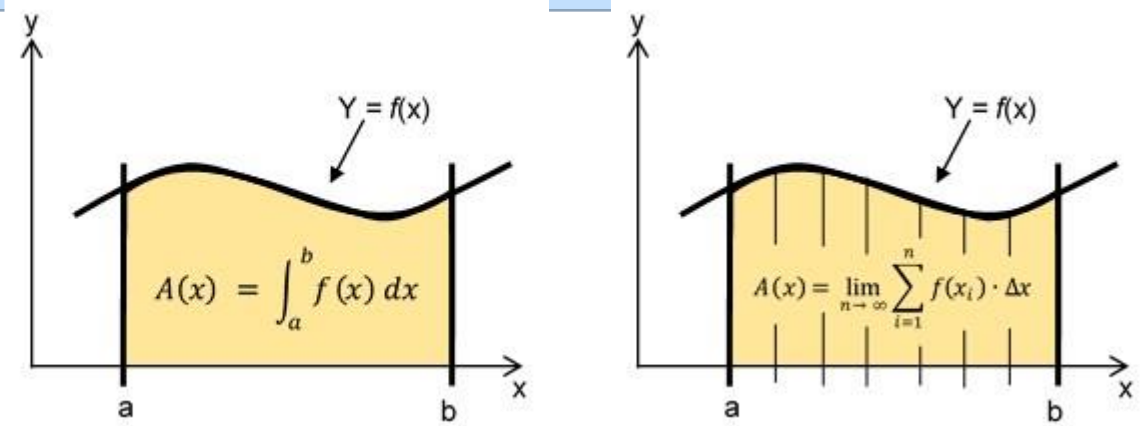
As you saw in section 5.3, the "**Definite Integral**" can be approximated with a Riemann sum (dividing the area of integration, into rectangles where the height of each rectangle comes from the function ($y = f(x)$), multiplied by the width (Δx), computing the area of each rectangle, and adding them up).

The more rectangles we use, the narrower the rectangles are, the better our approximation will be.

On the left side of the equation, we integrate, the definite integral of a function, which equals to the Riemann sum on the right side of the equation below.

Formal Definition of Definite Integral:

$$A(x) = \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$



The "**Definite Integral**" of a positive function ($f(x)$) over an interval [***a***, ***b***] is the "area" between (f), the x-axis interval, x=***a*** and x=***b***. The function (f) is called the integrand. The interval [***a***, ***b***] are called the limits of integration.

The ∫ symbol is called the integral sign, standing for sum. (The ∫ corresponds to the ∑ from the Riemann sum)

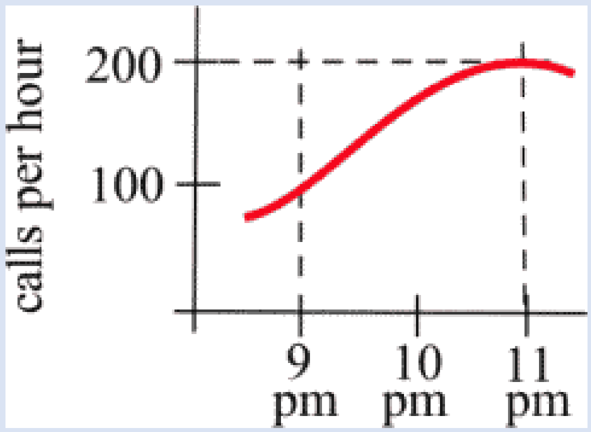
The (*d*x) on the end must be included! The dx tells what the variable is; the variable is x. The (*d*x) corresponds to the (Δx) from the Riemann sum).

Example Problem #1

The graph below shows the function ($y = r(t)$), the number of telephone calls made per hours on a Tuesday.

Approximately how many calls were made between 9 pm and 11 pm?

Express this as a definite integral and approximate with a Riemann sum.

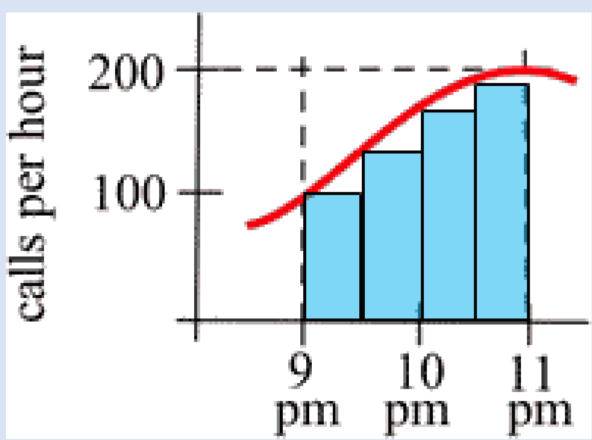


We know that the accumulated calls will be the area under this rate graph over that two-hour period; the definite integral of this rate from [t = 9 to t = 11].

The total number of calls will:

$$Total\ Number\ of\ calls \quad = \int_{9}^{11} r(t)\,dt$$

Next, we can approximate the area using rectangles. Let's choose to use four rectangles and left-endpoints:



Example Problem #1 – Cont'd

The graph above shows the function ($y = r(t) \rightarrow {}^{calls}/_{hr}$), the number of telephone calls made per hour.
The total number of calls is described by the following equation:

$$Total\ Number\ of\ calls \ = \int_{9}^{11} r(t)\,dt \ = \ \lim_{n \to 4} \sum_{i=1}^{4} r(t_i) \cdot \Delta t$$

Next, the rectangular width is the fixed "time interval" (Δt) calculated:

$$\Delta t \ = \ \frac{b\ -\ a}{n} \ = \ \frac{11\ -\ 9}{4} \ = \ \frac{2}{4} \ = \ 0.5$$

Next, the Riemann Sum is calculated, using the "**Left-Hand**" end points:

$$\sum_{i=1}^{4} r(t_i) \cdot \Delta t \ = r(t_1) \cdot \Delta t \ + r(t_2) \cdot \Delta t \ + r(t_3) \cdot \Delta t \ + r(t_4) \cdot \Delta t$$

Next, vertical values ($y = r(t)$)), are obtained from the graph:

$$\sum_{i=1}^{4} r(t_i) \cdot \Delta t \ = \ \left[\begin{array}{c} (100)^2 \cdot (0.5) \ + \ (150)^2 \cdot (0.5) \\ + (180)^2 \cdot (0.5) \ + \ (195)^2 \cdot (0.5) \end{array} \right] \ = \ 312.5\ Calls$$

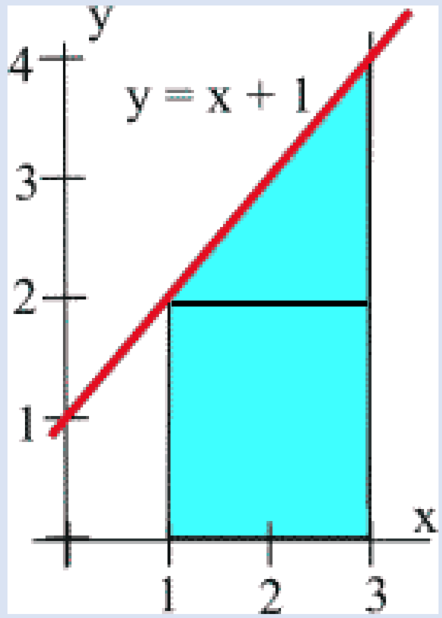
The total number of calls is:

$$\begin{pmatrix} Total\ Number \\ of\ Calls \end{pmatrix} \ = \int_{9}^{11} r(t)\,dt \ = \ \lim_{n \to 4} \sum_{i=1}^{4} r(t_i) \cdot \Delta t \ = \ 312.5\ Calls$$

Our estimate is that about 312 calls were made between 9 pm and 11 pm.

Example Problem #2

Using the Definite Integral and the idea of area, to determine the value of the function, ($y = f(x) = (1 + x)$); on the interval [x = 1 to x = 3].



$$Area = \int_1^3 f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$Area = \int_1^3 (1 + x)\,dx$$

Since this area can be broken into a rectangle and a triangle, we can find the area exactly. The total area equals:

$$Area = \int_1^3 (1 + x)\,dx = Square + Triangle$$

$$Area = \int_1^3 (1 + x)\,dx = (2)(2) + \frac{1}{2}(2)(2)$$

$$Area = \int_1^3 (1 + x)\,dx = 4 + 2 = 6\,units^2$$

The Fundamental Theorem of Integral Calculus

This section contains the most important and most used theorem of calculus, the Fundamental Theorem of Calculus.

Discovered independently by Newton and Leibniz in the late 1600s, it establishes the connection between derivatives and integrals, provides a way of easily calculating many integrals, and was a key step in the development of modern mathematics to support the rise of science and technology.

Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is a most important brick in that beautiful structure.

Fundamental Theorem of Integral Calculus:

$$F(x) = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

For a continuous function (f) on an interval [***a***, ***b***]
Where (F) is any antiderivative of (f), hence (f) is the derivative of (F).

The endpoints of the interval [***a***, ***b***] are known as the limits of integration.

A function followed by a vertical bar ($\Big|_a^b$), listed with the limits of integration numbers [***a*** and ***b***] means to evaluate the function at the upper limit number (***b***), and then subtract the evaluation at the lower limit number (***a***).

The definite integral of a rate from ***a*** to ***b*** is the net y-units, the change in y, that accumulate between x=***a*** and x=***b***. The rate is a derivative.

We will not give a proof of this Theorem in this text, but there are many resources on the web for this topic. Here is the link to one of them:

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-7/a/proof-of-fundamental-theorem-of-calculus

Example Problem #3

Using the Fundamental Theorem of Integral Calculus or the Definite Integral to find the area under the function ($f(x) = x^2$), on interval from 1 to 3.

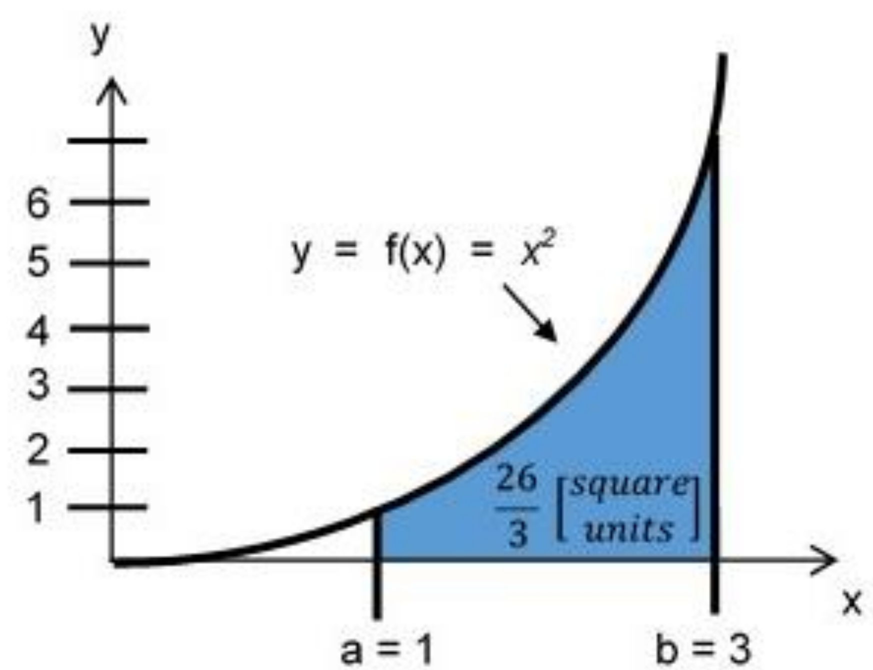
Solution:

Let $a = 1$, and, $b = 3$

$$F(x) = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$F(x) = \int_1^3 x^2 dx = \left[\frac{1}{2+1}\cdot x^{2+1}\right]_{x=1}^{x=3}$$

$$F(x) = \frac{1}{3}\cdot x^3\Big|_1^3 = \frac{1}{3}\cdot(3)^3 - \frac{1}{3}\cdot(1)^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}\ units^2$$



Example Problem #4

Find the "area" under the curve of the function ($y = f(x) = e^{2x}$), on the interval from x = 0 to x = 2.

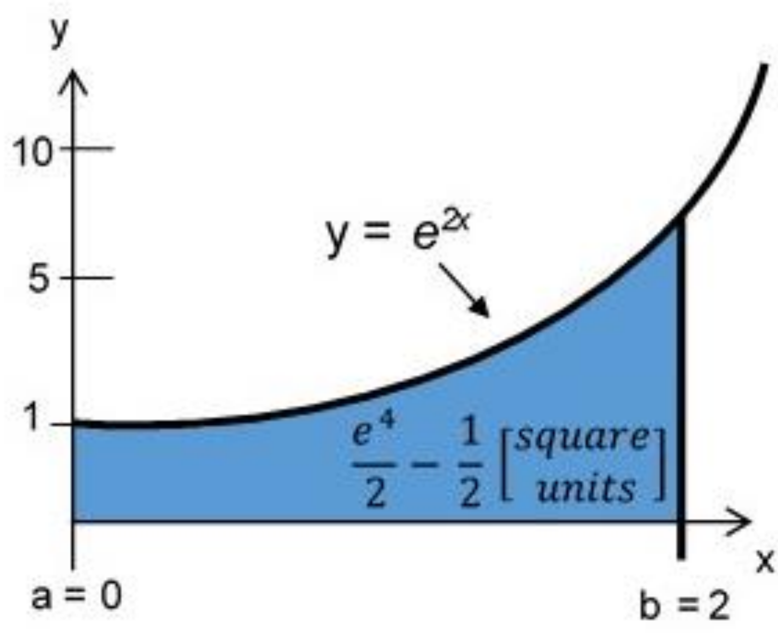
Solution:

Let $a = 0$, and, $b = 2$

$$F(x) = \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$F(x) = \int_0^2 e^{2x} dx = \left[\frac{1}{2} \cdot e^{2x}\right]_{x=0}^{x=2}$$

$$F(x) = \frac{1}{2} \cdot (e)^{2(2)} - \frac{1}{2} \cdot (e)^{2(0)} = \frac{e^4}{2} - \frac{1}{2} = \frac{e^4 - 1}{2} \ units^2$$



Example Problem #5

Find the "area" under the curve of the function ($y = f(x) = \frac{1}{x}$), on the interval from x = 1 to x = e.

Solution:

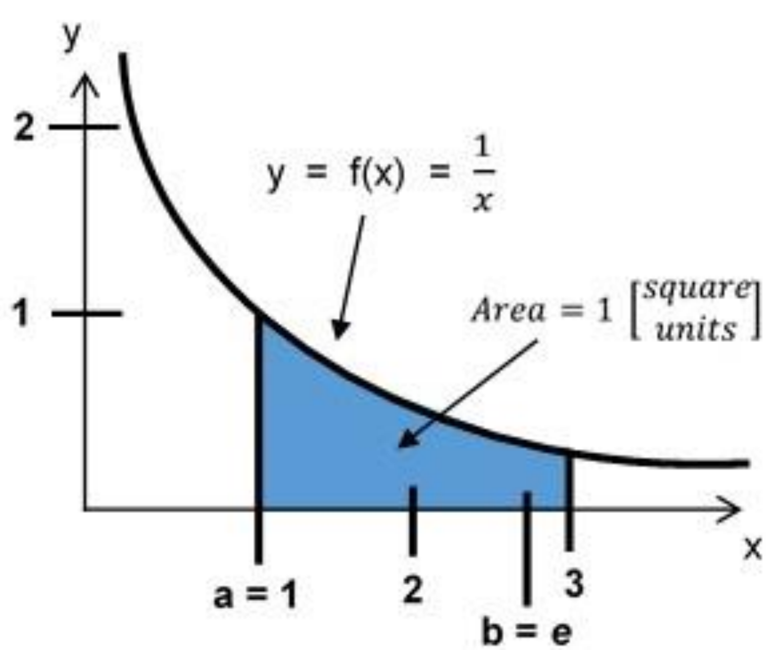
Let $a = 1$, and, $b = e$

$$F(x) = \int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

$$F(x) = \int_1^e \frac{1}{x}dx = ln(x)|_1^e$$

$$F(x) = \ln(e) - \ln(1) = 1\ units^2$$

Therefore, the area is 1 square unit.



Business Cases using Definite Integration

Given a "**Marginal Cost**" ($MC(x)$) function we can find the "**Total Accumulated Cost**" ($C(x)$) of producing say, between a and b many units.

In the same manner, given the "**Marginal Revenue**" ($MR(x)$) function we can find the "**Total Accumulated Revenue**" ($R(x)$) from selling say, between a and b many units.

Furthermore, the "**Marginal Profit**" ($MP(x)$) function we can find the "**Total Accumulated Profit**" ($P(x)$) from selling say, between a and b many units.

The **"Marginal Profit"** ($MP(x) = MR(x) - MC(x)$) is also the difference between the Marginal Revenue and the Marginal Cost.

Total Accumulated Cost, Revenue and Profit.

$\boldsymbol{Total\ Cost}\ of\ producing\ a\ to\ b\ many\ units$

$$C(x) = \int_a^b MC(x)\,dx$$

$\boldsymbol{Total\ Reneue}\ from\ selling\ a\ to\ b\ many\ units$

$$R(x) = \int_a^b MR(x)\,dx$$

$\boldsymbol{Total\ Profit}\ made\ from\ producing\ aand\ selling\ to\ b\ many\ units$

$$P(x) = \int_a^b MP(x)\,dx = \int_a^b (MR(x) - MC(x))\,dx$$

$$P(x) = \int_a^b MR(x)\,dx - \int_a^b MC(x)\,dx$$

$$P(x) = R(x) - C(x)$$

Example Problem #6

A company's "**Marginal Profit**" ($MP(x)$), where (x) is the number of units is:

$$MP(x) = 80\sqrt[3]{x} \quad \text{dollars/unit}$$

Find the "**Total Profit**" $P(x)$ gained from producing and selling the first 8000 units.

Solution:

Since we are talking about the first 8000 units, a = 0 and b = 8000.

$$Total\ profit = P(x) = \int_a^b MP(x)\,dx$$

$$P(x) = \int_0^{8000} 80\sqrt[3]{x}\ dx = 80\int_0^{8000} x^{\frac{1}{3}}\,dx$$

$$P(x) = 80\left[\frac{1}{\frac{1}{3}+1}\cdot x^{\frac{1}{3}+1}\right]_0^{8000}$$

$$P(x) = 60\,x^{\frac{4}{3}}\Big|_0^{8000} = 60\cdot\left(\sqrt[3]{8000}^{\,4} - \sqrt[3]{0}^{\,4}\right)$$

$$P(x) = 60\cdot(160{,}000 - 0) = 9{,}600{,}000 \text{ dollars}$$

The "**Total accumulated profit**" ($P(x)$) from producing and selling the first 8000 units is $\$\ 9{,}600{,}000$.

Definite Integrals have the same properties as the indefinite integrals we discussed in section 5.1. These properties will allow us to find **Definite Integrals** of more complicated functions.

Constant Multiple - Rule of Definite Integrals:

$$\int_a^b K \cdot f(x)\,dx = K \cdot \int_a^b f(x)\,dx$$

Addition/Subtraction - Rule of Definite Integrals:

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Example Problem #7

Using the properties of "Definite Integrals"; Find the "Total Area" under the curve, for the function ($y = f(x)$), on the interval from $x = -2 \; to \; x = 2$.

$$y = f(x) = 36x^2 + 18x - 6$$

Solution:

$$F(x) = \int_{-2}^{2} \left(36x^2 + 18x - 6 \right) dx$$

$$F(x) = 36\int_{-2}^{2} x^2 dx + 18\int_{-2}^{2} x dx - 6\int_{-2}^{2} 1 dx$$

$$F(x) = 36\left[\frac{1}{3}x^3\right]_{-2}^{2} + 18\left[\frac{1}{2}x^2\right]_{-2}^{2} - 6[x]_{-2}^{2}$$

$$F(x) = 12\left[2^3 - (-2)^3\right] + 9\left[2^2 - (-2)^2\right] - 6[2 - (-2)]$$

$$F(x) = 12[16] + 9[0] - 6[4] = 192 - 24 = 168 \; unit^2$$

The "Total Area" under the curve from $\boldsymbol{x = -2 \; to \; x = 2}$ is **168 square units**.

5.4 - EXERCISES

Use a ***definite Integral*** to find the area under the curve of the given functions between the given a and b values.			
1.	$f(x) = x^3$ from ***a*** = 0 to ***b*** = 2	2.	$f(x) = 8x^3 + 5$ from ***a*** = 1 to ***b*** = 4
3.	$f(x) = x^2 + 8$ from ***a*** = -3 to ***b*** = 3	4.	$f(x) = \frac{1}{x}$ from ***a*** = 2 to ***b*** = 5
5.	$f(y) = \frac{1}{y^2}$ from ***a*** = 1 to ***b*** = 5	6.	$f(x) = \frac{1}{4}e^{\frac{x}{4}}$ from ***a*** = 0 to ***b*** = 24
7.	$f(x) = 15x^2 + 8x - 6$ from ***a*** = 0 to ***b*** = 2		

Evaluate each ***Definite Integral***			
8.	$\int_1^5 \frac{1}{x^2}\, dx$	9.	$\int_0^1 2e^2 t\, dt$
10.	$\int_3^4 (3 + 16x^{-3})\, dx$	11.	$\int_1^2 (25x^2 + 5x^{-2})\, dx$
12.	$\int_0^2 (12x^2 + 16e^{4x})\, dx$	13.	$\int_1^3 \left(48x^3 - \frac{4}{x}\right) dx$
14.	$\int_1^{32} \left(\frac{1}{\sqrt[5]{x^2}} + 3\right) dx$	15.	$\int_0^1 (25x^{24} + 12x^{11} + 2)\, dx$

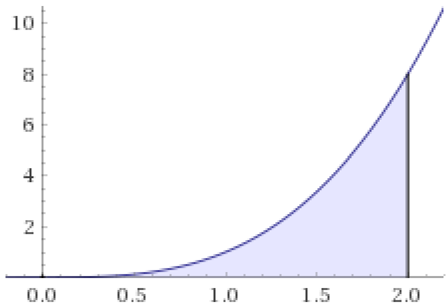
16.	In general, the purchase price of an electronic device will decrease over time. In 2015, the price of an LCD Large Screen TV was $3500. Since 2015, the price of an LCD Large Screen TVs is decreasing at the rate of ($g(t)$) hundred dollars per year, where (t) is measured in years and ($t = 0$) corresponds to 2015. Find the total change in price between the years 2015 and 2020. $$g(t) = 3500e^{-\frac{6}{5}t}$$
17.	A Tire Company can produce tires at a steady rate ($T(x)$) units per hour, where (x) is the number of hours of work past the starting time of 8:00 am. ($0 \le x \le 8$). Find the number of tires produced by the company between the hours of 10:00 a.m. and 3:00 p.m. (5 hours)? $$T(x) = 90x^2 + 60x + 20$$
18.	A farmer can pick oranges at the rate of ($v(t)$) oranges per minute after (t) minutes. Find the total number of oranges, that the farmer can pick, over a time interval between 9:00 a.m. and 9:12 am? $$v(t) = 600e^{-6t}$$
19.	A good math professor can grade a large class of exams, at the rate of ($r(t)$) exams per hour. How many exams will the math professor grade during the first two hours. (time 0 to time 2)? $$r(t) = -3t^2 + 18t + 4$$

20.	A company's Marginal Cost ($MC(x)$), where (x) is the number of units is: $$MC(x) = 15\sqrt{x}$$ dollars/unit Find the cost of producing between 100 and 900 units.

Solutions

1. $4\ square\ units$

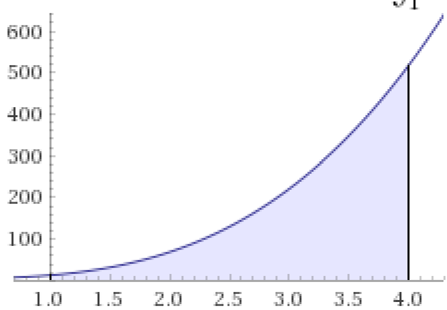
$$\int_0^2 x^3\,dx = 4$$



2. $525\ square\ units$

$$\int (8x^3+5)\,dx = 2x^4 + 5x + \text{constant}$$

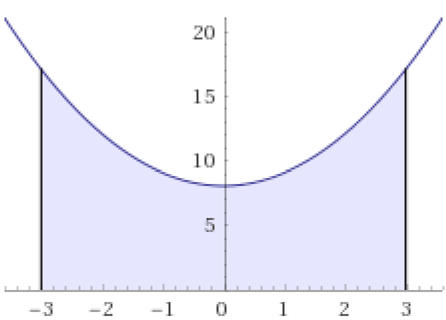
$$\int_1^4 (8x^3+5)\,dx = 525$$



3. $66\ square\ units$

$$\int (x^2+8)\,dx = \frac{x^3}{3} + 8x + \text{constant}$$

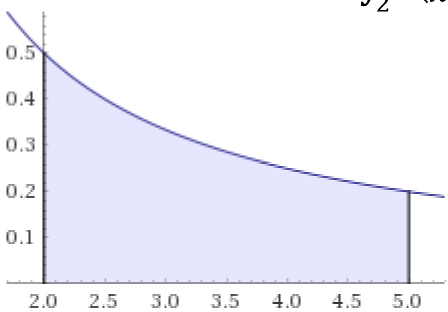
$$\int_{-3}^3 (x^2+8)\,dx = 66$$



4. $0.9163\ square\ units$

$$\int \left(\frac{1}{x}\right) dx = \ln\left(\frac{5}{2}\right) + C$$

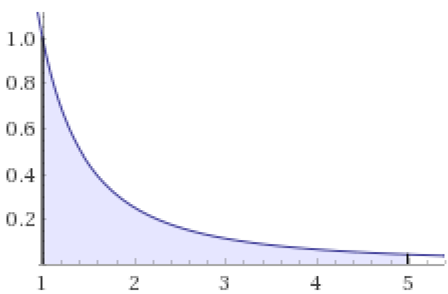
$$\int_2^5 \left(\frac{1}{x}\right) dx = \ln\left(\frac{5}{2}\right) \approx 0.9163$$



5. $-0.8\ square\ units$

$$\int \frac{1}{y^2}\, dy = -\frac{1}{y} + \text{constant}$$

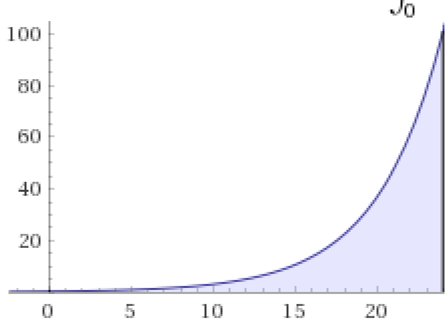
$$\int_1^5 \frac{1}{y^2}\, dy = \frac{4}{5} = 0.8$$



6. $402.43\ square\ units$

$$\int \frac{e^{x/4}}{4}\, dx = e^{x/4} + \text{constant}$$

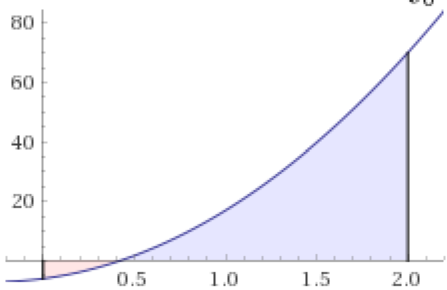
$$\int_0^{24} \frac{e^{x/4}}{4}\, dx = e^6 - 1 \approx 402.43$$



7. $44 \; square \; units$

$$\int (15x^2 + 8x - 6)\,dx = 5x^3 + 4x^2 - 6x + \text{constant}$$

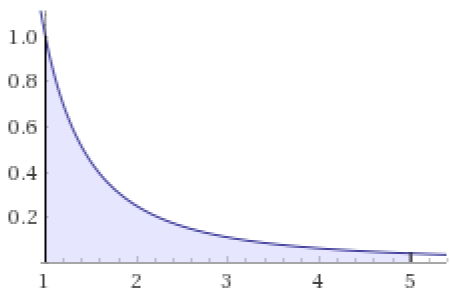
$$\int_0^2 (15x^2 + 8x - 6)\,dx = 44$$



8. 0.8

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + \text{constant}$$

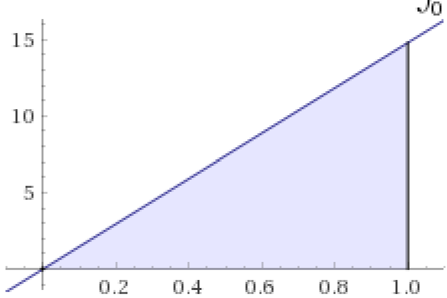
$$\int_1^5 \frac{1}{x^2}\,dx = \frac{4}{5} = 0.8$$



9. 7.389

$$\int 2e^2 t\,dt = e^2 t^2 + \text{constant}$$

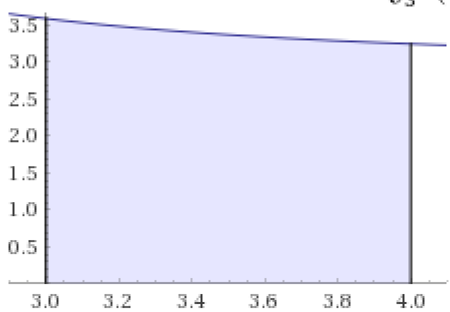
$$\int_0^1 2e^2 t\,dt = e^2 \approx 7.3891$$



10. $\frac{61}{18} \approx 3.39$

$$\int \left(3 + \frac{16}{x^3}\right) dx = 3x - \frac{8}{x^2} + \text{constant}$$

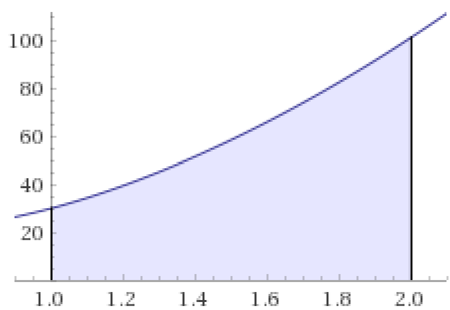
$$\int_3^4 \left(3 + \frac{16}{x^3}\right) dx = \frac{61}{18} \approx 3.3889$$



11. $\frac{365}{6} = 60.833$

$$\int \left(25x^2 + \frac{5}{x^2}\right) dx = \frac{25x^3}{3} - \frac{5}{x} + \text{constant}$$

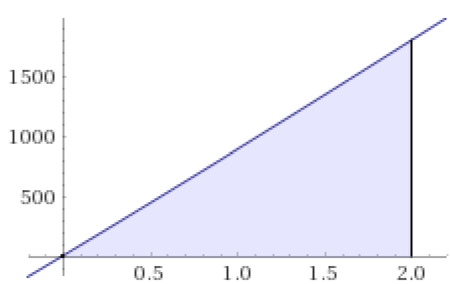
$$\int_1^2 \left(25x^2 + \frac{5}{x^2}\right) dx = \frac{365}{6} \approx 60.833$$



12. 11952

$$\int (12x^2 + 16e^{4x})\, dx = 4(x^3 + e^{4x}) + \text{constant}$$

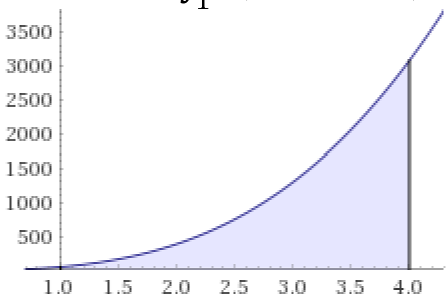
$$\int_0^2 (12x^2 + 16e^{4x})\, dx = 4(7 + e^8) \approx 11\,952.$$



13. 955.61

$$\int \left(48x^3 - \frac{4}{x}\right) dx = 12x^4 - 4\ln(x) + C$$

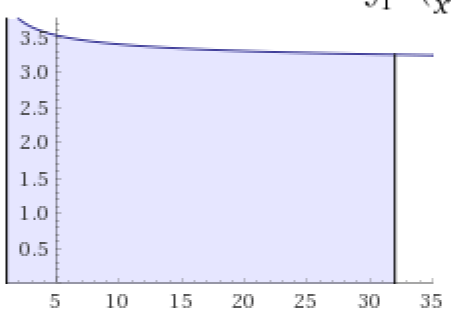
$$\int_1^3 \left(48x^3 - \frac{4}{x}\right) dx = 960 - 4\ln(3) \approx 955.61$$



14. 104.67

$$\int \left(\frac{1}{x^{2/5}} + 3\right) dx = \frac{5\,x^{3/5}}{3} + 3\,x + \text{constant}$$

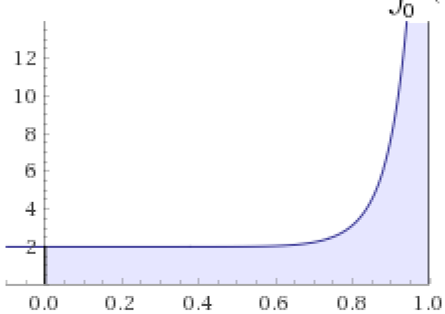
$$\int_1^{32} \left(\frac{1}{x^{2/5}} + 3\right) dx = \frac{314}{3} \approx 104.67$$



15. 4

$$\int (25\,x^{24} + 12\,x^{11} + 2)\,dx = x^{25} + x^{12} + 2\,x + \text{constant}$$

$$\int_0^1 (25\,x^{24} + 12\,x^{11} + 2)\,dx = 4$$



16. $\$2909.44$ $-$ *Amount of price decline in LCD Big Screen TVs*

$$from: 2015 - 2020$$

$$\int 3500\, e^{-1.2t}\, dt = -\frac{8750}{3}\, e^{-(6t)/5} + \text{constant}$$

$$\int_0^5 3500\, e^{-1.2t}\, dt = 2909.44$$

17. $11{,}500\ Tires$

$$11{,}500\ Tires\ produced\ in\ five\ (5)\ hours,$$
$$between\ the\ hours\ of\ 10{:}00\ am - 3{:}00\ pm$$

18. $100\ Oranges$

$$100\ Oranges\ in\ 12\ minutes$$
$$in\ the\ interval\ 9{:}00\ am - 9{:}12\ am$$

$$\int 600\, e^{-6t}\, dt = -100\, e^{-6t} + \text{constant}$$

$$\int_0^{12} 600\, e^{-6t}\, dt = 100 - \frac{100}{e^{72}} \approx 100.00$$

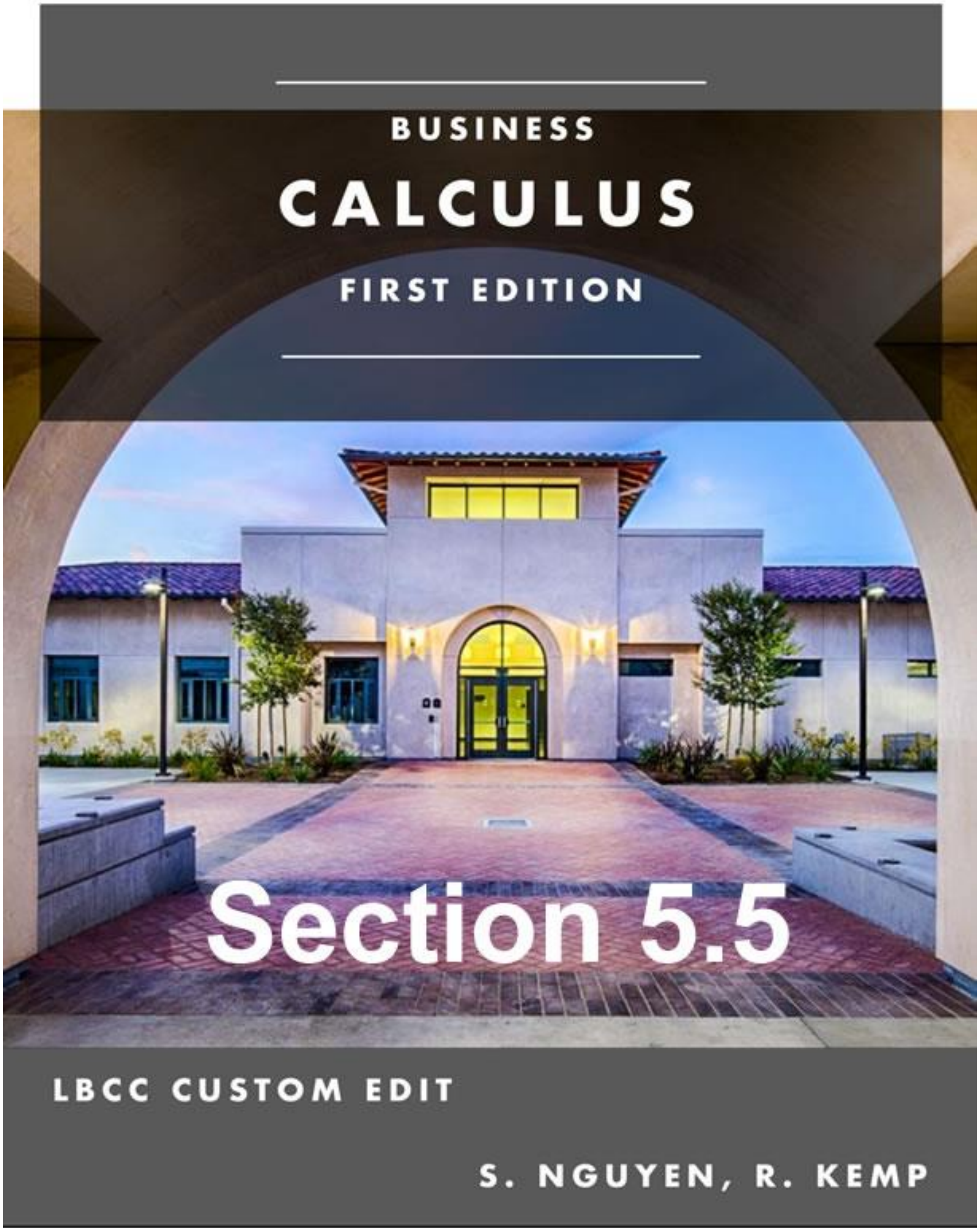
19. $36\ Graded\ Exams - in\ two\ hours$

$$\int (-3t^2 + 18t + 4)\, dt = -t^3 + 9t^2 + 4t + \text{constant}$$

$$\int_0^2 (-3t^2 + 18t + 4)\, dt = 36$$

20.

$\$260{,}000\ is\ the\ total\ cost\ of\ producing\ between\ 100\ and\ 900\ units.$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5.5 - AVERAGE VALUE – DEFINITE INTEGRATION

Introduction

What if we need to find the average temperature over a day's time – there are too many possible temperatures to add them up! This is a job for the definite integral.

Averages eliminate fluctuations, reducing a collection of numbers to a single representative number.

Average Value of a Function

We know the average (f_{Avg}) value, of (n) numbers $[a_1, a_2, a_3, a_1, \dots . a_n]$, is found by adding all the numbers, and dividing by (n); for example:

$$f_{Avg} = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

If a function gives the curve for the temperature over a (24 hour) time-period, we could determine the average value of the temperature, by measuring the value of the temperature every hour, then average the 24 different values.

However, this would ignore the temperature at all the intermediate times, i.e.: 30-minute, 15-minute, or 5-minute intervals.

Intuitively, the average valued represents a value that is mid-way (average) between the highest peak temperature, and the lowest peak temperature over the (24 hour) period, curve.

Finding the "Average" value of a function also has the advantage of considering all the values of the function in the interval.

Therefore, the "average" area under the cure, is given by a "Rectangular Area" $(A_{Area} = HL)$, whose height $(H = H_{Avg})$, and base is $(L = b - a)$.

By the "Mean Value Theorem" of Calculus.

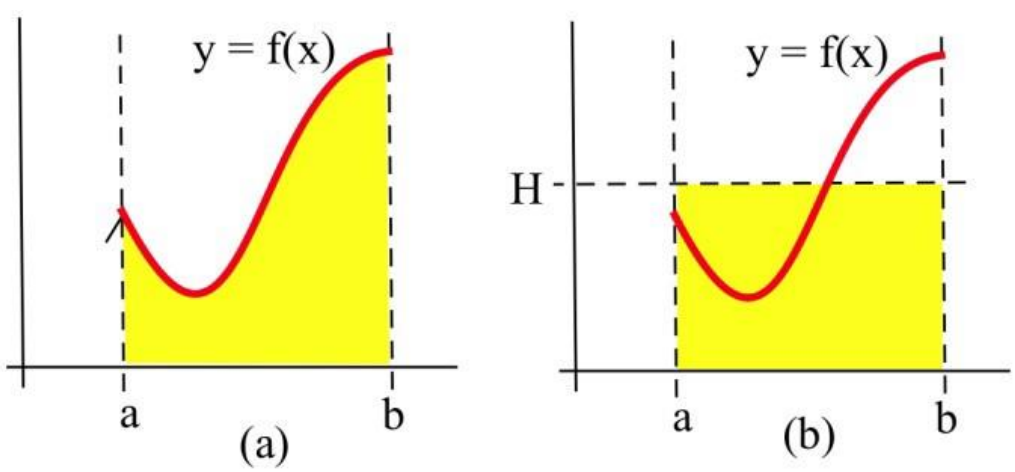
$$A_{Area} \;=\; H_{Avg} \cdot L \;=\; H(x)_{Avg} \cdot (b-a)$$

$$H(x)_{Avg} \cdot (b-a) \;=\; \int_a^b f(x)\,dx \;=\; F(b) \;-\; F(a)$$

The formula that gives the "Average" or "Mean" value of a continuous function on an interval from [***a*** to ***b***].

Average Value of a Function

$$H(x)_{Avg} \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx \;=\; \frac{F(b) \;-\; F(a)}{b-a}$$



The "Average" value of a function (f), is the average height (H_{Avg}) of the rectangle whose area is the same as the area under the curve (f), on an interval from [***a*** to ***b***].

Finding the "Average" value of a function (f), by integrating and dividing by the base interval ($b-a$), is the same as computing a Riemann Sum, which would be the averaging (n) numbers by adding and dividing by (n); because integrals are continuous sums.

Average Value of a Function – Additional Description

In general, the average value of a function will have the same units as the integrand.

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals.

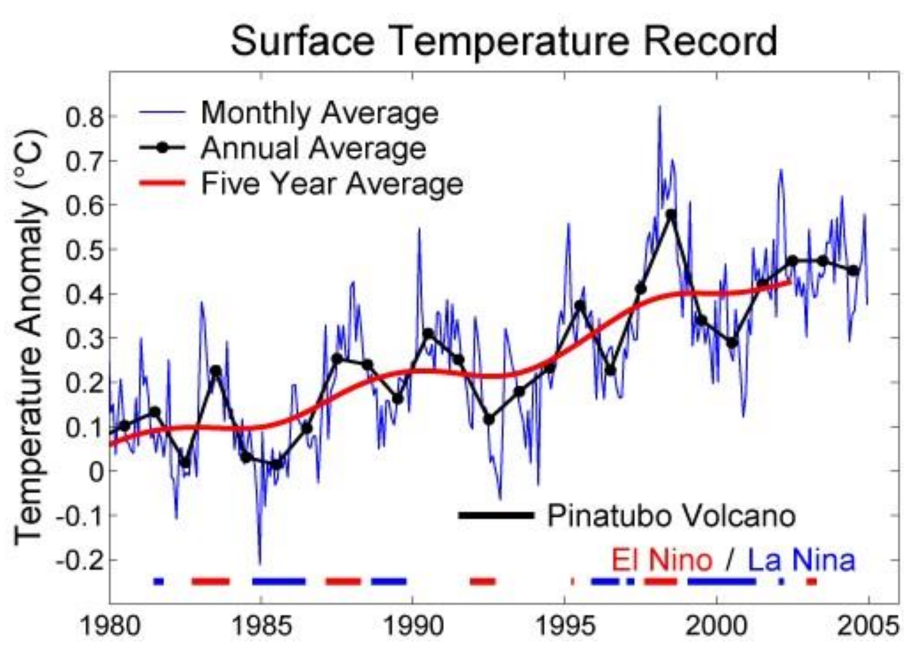
In these situations, the original function ($f(x)$) is replaced by some "average" ($g(x)_{Avg}$). If ($f(x)$), is rather jagged time data, then the ten-year average of ($g(x)_{Avg}$), is the integral:

$$g(x)_{Avg} = \frac{1}{10}\int_{x-5}^{x+5} f(t)\, dt$$

an average of ($f(x)_{Avg}$), over 5 units on each side of (x).

For example, the figure below[1], shows the graphs of:

- Monthly Average (rather "noisy" data) of surface temperature data.
- An Annual Average (still rather "jagged), and;
- A Five-Year Average (a much smoother function).



[1] http://commons.wikimedia.org/wiki/File:Short_Instrumental_Temperature_Record.png, CC-BY

Typically, the average function reveals the pattern much more clearly than the original data. This use of a "moving average" value of "noisy" data (weather information, traffic information, or stock prices) is very common.

Example Problem #1

During a (9 hour) workday, the production rate at time (t) hours, after the start of the shift, was given by the function $(r(t))$, cars per hour.

Find the average hourly production rate?

$$r(t) = \left(5 + \sqrt{t}\right) \ cars/hr$$

Solution:

The average hourly production is:

$$r(t)_{Avg} = \frac{1}{b-a}\int_a^b r(t)\, dt$$

an interval from [***a* = *0*** to ***b* = *9***].

$$r(t)_{Avg} = \frac{1}{9-0}\int_0^9 \left(5 + \sqrt{t}\right) dt$$

$$r(t)_{Avg} = \frac{1}{9}\left[5t + \frac{2}{3}\left(t^{\frac{3}{2}}\right)\right]_0^9$$

$$r(t)_{Avg} = \frac{1}{9}\left[\left(5(9) + \frac{2}{3}\left((3^2)^{\frac{3}{2}}\right)\right) - (0 + 0)\right]$$

$$r(t)_{Avg} = \frac{1}{9}[45 + 18] = \frac{1}{9}[63] = 7$$

$$r(t)_{Avg} = 7 \ Cars/hr$$

A note about the units – remember that the definite integral has units.

(cars per hour) • (hour) = Cars.

But the $(\frac{1}{b-a})$ in front has units $(\frac{1}{Hours})$ the units of the average value, are cars per hour, just what we expect an average rate to be.

Example Problem #2

Find the Average Value of the function:

$$f(x) = 72x^2 \quad \text{from} \quad x = 1 \text{ to } x = 5$$

Solution:

The Average Value is:

$$f(x)_{Avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

an interval from [***a* = 1** to ***b* = 5**].

$$f(x)_{Avg} = \frac{1}{5-1}\int_1^5 72x^2\,dx = \frac{72}{4}\int_1^5 x^2\,dx$$

$$f(x)_{Avg} = 18\left[\frac{x^3}{3}\right]_1^5 = 6[x^3]_1^5$$

$$f(x)_{Avg} = 6[(5)^3 - (1)^3] = 6 \cdot (125 - 1) = 6 \cdot (124)$$

$$f(x)_{Avg} = 744$$

The Average Value of ($f(x) = 72x^2$), over the interval [1, 5], is **744**.

Example Problem #3

Finding the Average Population.
Based on the Census from year 2010 data, California's population was about 37.25 million people; and is growing exponentially according to the function ($P(t)$).

Let (t) be the number of years since 2010.

Predict the Average population of California between the years 2020 and 2025?
Source: census.gov

$$P(t) = 37.25e^{0.0065t}$$

Solution:
We set the initial start time at 2010 to equal zero: [t = 0 (year 2010)]

We integrate from [t = 10 (year 2020)] to [t = 15 (year 2025)].

$$P(t)_{Avg} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$P(t)_{Avg} = \frac{1}{15-10}\int_{10}^{15} 37.25e^{0.0065t}\,dt$$

$$P(t)_{Avg} = 7.45\int_{10}^{15} e^{0.0065t}\,dt$$

$$P(t)_{Avg} = 7.45\left[\frac{e^{0.0065t}}{0.0065}\right]_{10}^{15} = 1146.15[e^{0.0065t}]_{10}^{15}$$

$$P(t)_{Avg} = 1146.15\left[e^{0.0065(15)} - e^{0.0065(10)}\right]$$

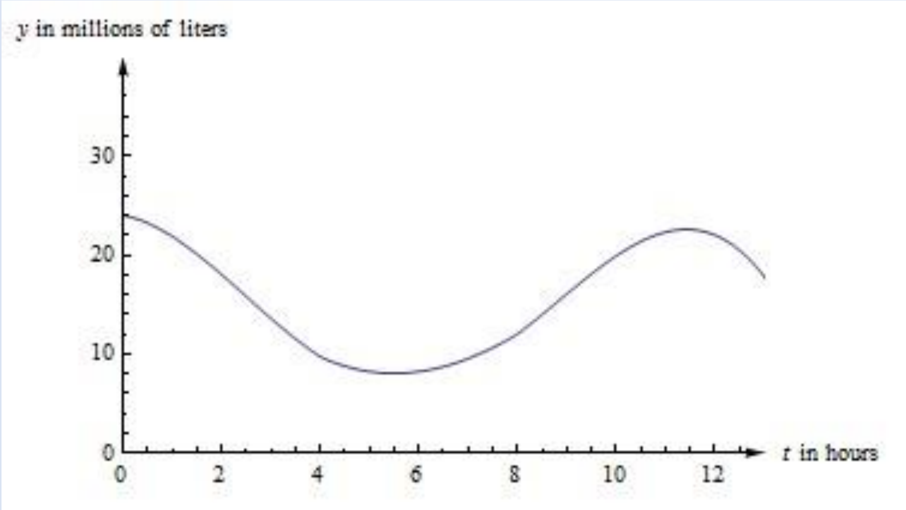
$$P(t)_{Avg} \cong 1146.15[1.1024 - 1.0672] \cong 40.3445$$

The Average population of California between the years 2020 and 2025, will grow to be about 40.3445 million people, or about 40,344,500 people.

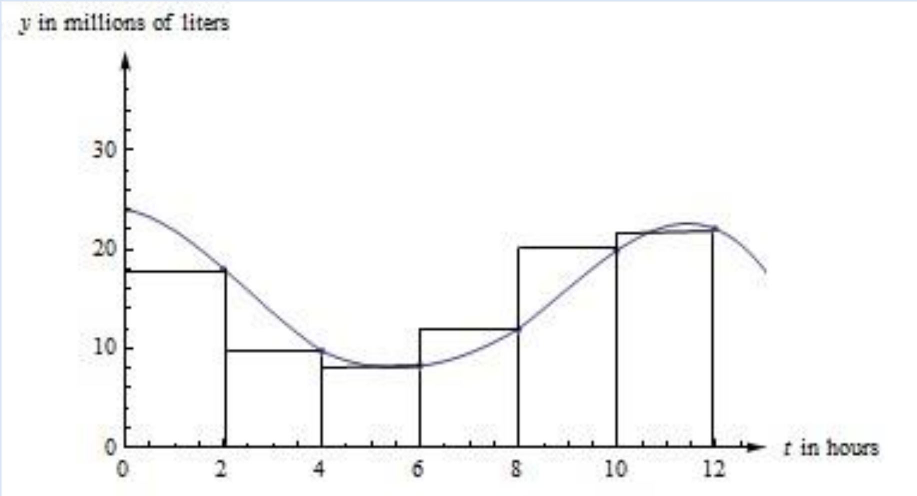
Example Problem #4

Estimate the average amount of water.
The graph below shows the amount of water in a reservoir over a (12 hour) period. Estimate the average amount of water in the reservoir over this period.

If $(V(t))$ then the average



To find the definite integral, we'll have to estimate. Let's use (n = 6) rectangles and take the heights from their right edges (there's nothing special about using 6 rectangles or right edges – other choices would still give you a valid estimate).



Example Problem #4 – Cont'd

The estimate of the integral is

$$V(t)_{Avg} = \frac{1}{12}\int_0^{12} V(t)\,dt$$

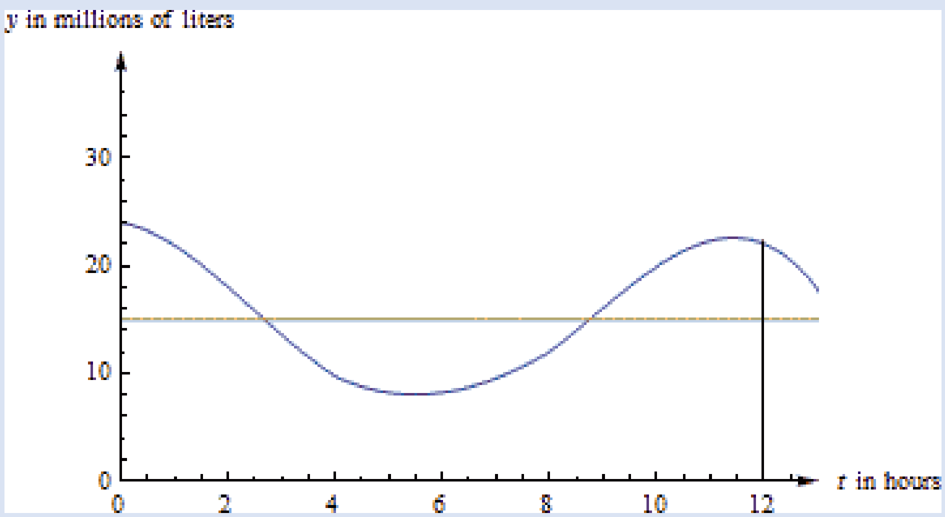
$$V(t)_{Avg} = \frac{1}{12}\begin{bmatrix} (18)(2) + (9.7)(2) + (8.2)(2) \\ + (12)(2) + (19.9)(2) + (22)(2) \end{bmatrix}$$

$$V(t)_{Avg} = \frac{179.6}{12} \cong 15$$

The units of this integral are (Millions of Liters) · (Hours).
The estimate of the average volume is:

$$V(t)_{Avg} = \frac{179.6}{12} \cong 15 \textit{ Million Liters}$$

The estimate might change a little depending on how we estimate the function values from the graph. In the figure below, you can see the same graph with the line (**y = 15**) drawn in.



The area under the curve and the area under the rectangle are (approximately) the same. In fact, that would be a different way to estimate the average value.

We could have estimated the placement of the horizontal line so that the area under the curve and under the line were equal.

5.5 - EXERCISES

Find the Average Value of each function over the given interval.	
1.	$f(x) = 4x - 5 \quad on\ [1, 3]$
2.	$f(x) = 9x^2 \quad on\ [0, 5]$
3.	$f(y) = \dfrac{1}{y^3} \quad on\ [-1, 7]$
4.	$f(x) = 27\sqrt{x} \quad on\ [0, 9]$
5.	$f(y) = 6y^2 - 4y \quad on\ [-1, 2]$
6.	$f(x) = 8 \quad on\ [15, 65]$
7.	$f(z) = -z^3 + 72 \quad on\ [-2, 2]$
8.	$g(x) = 8e^{\frac{x}{4}} \quad on\ [0, 4]$
9.	$g(z) = 4e^{0.05z} \quad on\ [0, 40]$
10.	$f(t) = \dfrac{5}{t} \quad on\ [1, 5]$

11.	A Grocery store's daily sales from produce, on day (t) are given by the function ($f(t)$) below. Find the average sales during the first 4 days (day 0 to day 4). $$f(t) = 4t^2 + \frac{14}{3}t$$
12.	An investor goes to a bank and makes a deposit of \$2250 at 7.5% interest which when compounded continuously, will grow to ($P(t)$) dollars after (t) years. Find the average value during the first 15 years (that is, from time 0 to time 15). $$P(t) = 2250e^{0.075t}$$
13.	A new Flying Car taxi service, is forecasting Profits to be ($P(x)$) million dollars per year, where (x) is the number of years since 2019. Predict the average annual profit from 2019 to 2024. $$P(x) = 45x^2 + 310x + 610$$

Solutions

41. 3

42. 75

43. $\frac{3}{49}$

44. 54

45. 4

46. 8

47. 72

48. $8(e - 1) = 13.746$

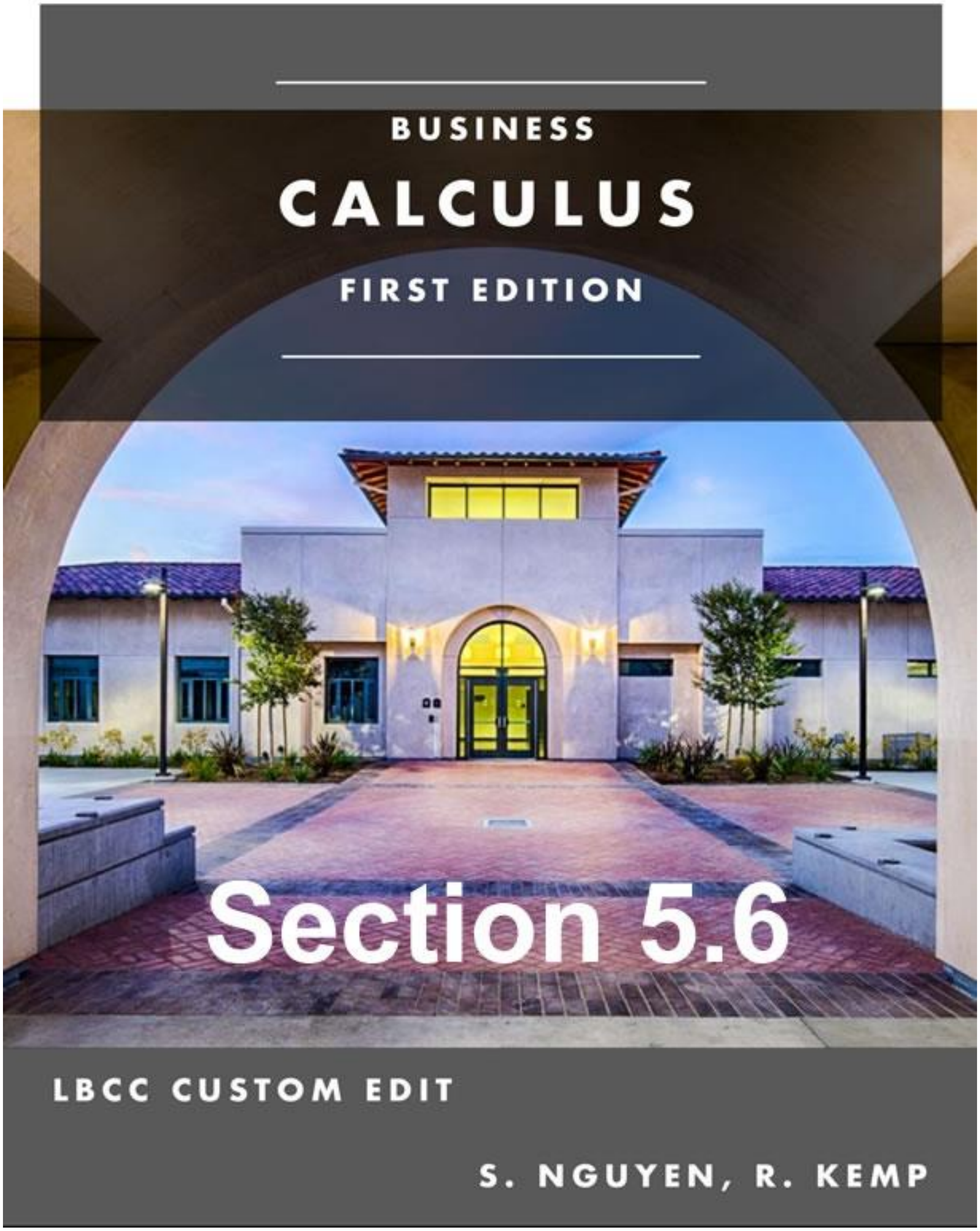
49. $2(e^2 - 1) = 12.778$

50. $1.25 \ln(5)$

51. $\frac{92}{3} = 30.66666 \rightarrow \$30.66$

52. $\$4{,}160.43$

53. $\$1.76\ billion\ (from\ \$1{,}760\ million)$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5.6 - AREA BETWEEN TWO CURVES – DEFINITE INTEGRATION

Introduction

The nature of Integral Calculus is to use integrals to find the “Area” beneath a curve, or between the graph of a function and the horizontal axis.

Integrals can also be used to find the “Area” between two curve or graphs.

The formula that gives the “Area” between two continuous “Curves” on an interval from [***a*** to ***b***], is given below.

Area Between Curves

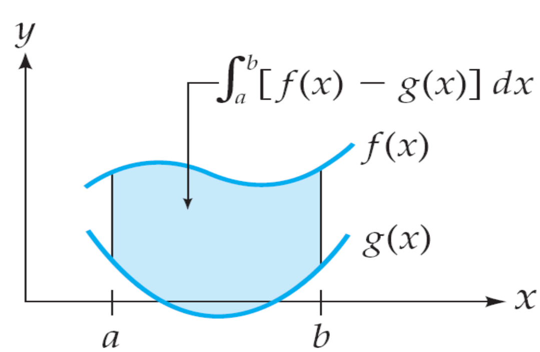
The area between two curves ($f(x)$ & $g(x)$),

Where ($f(x) \geq g(x)$) , [$x = a$] and [$x = b$]:

$$A_{Area} = \int_a^b [f(x) - g(x)]\, dx$$

The integrand is [**Top** “Upper” – **Bottom** “Lower”] = $[f(x) - g(x)]$.

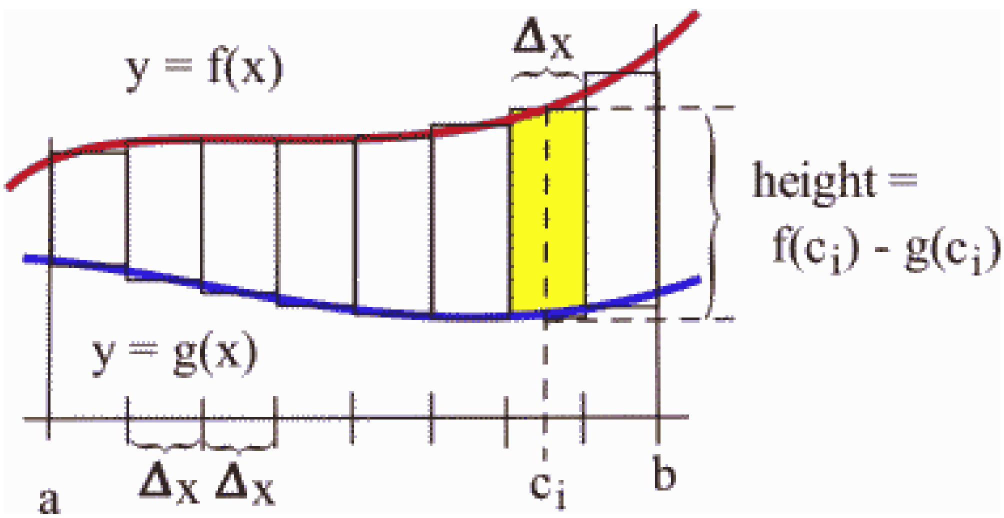
Finding area by integrating “upper minus lower” works regardless of whether one or both curves dip below the x- axis. Therefore, you should make a graph or use test values, to be sure which curve is the upper (top) and which on is the lower (bottom).



Area Between Curves – Integrating "Upper" Minus "Lower" Area

The Definite Integral predicts the "Area" beneath the curve. We can also use the Definite Integral to predict the "Area" between any two curves; and mathematically is equal to "Area" under the "**Top** - **Upper Curve**" subtracted from the "Area" under the "**Bottom** - **Lower Curve**"

If ($f(x) \geq g(x)$), for all (x) in the interval [a, b], then we can approximate the area between (f) and (g) by partitioning the interval [***a***, ***b***] and forming a Riemann sum, with (n) rectangles as shown in the picture.



The (**height**) of each rectangle is equal to: [**Top** – **Bottom**] = $[f(c_i) - g(c_i)]$

And the (**base**) of each rectangle is equal to:

$$\Delta x = \frac{b - a}{n}$$

And the "Area" of the (i-th) rectangle is (height)·(base):

$$A_{Area_i} = (height)(base) = [f(c_i) - g(c_i)] \cdot \Delta x$$

Adding up these rectangles gives an approximation of the "Total Area" as a Riemann sum:

$$A_{Area} = (height)(base\,) = \sum_{i=1}^{n} [f(c_i) - g(c_i)] \cdot \Delta x$$

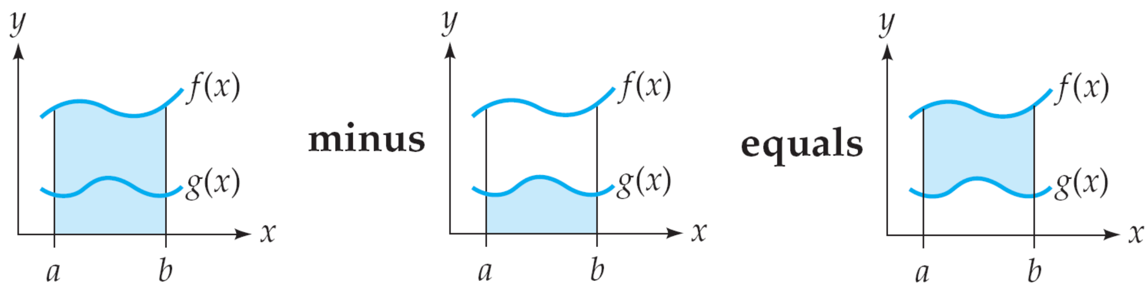
The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is the definite integral:

$$A_{Area} = \int_a^b [f(x) - g(x)]\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)] \cdot \Delta x$$

Area Bounded Between Curves

We know that definite integrals give areas under curves.

To calculate the area between two curves, we subtract the area with the "**Upper/Larger**" value from the area with the "**Lower/Smaller**" value, when integrating over a specific ($\boldsymbol{x-axis}$) or ($\boldsymbol{y-axis}$).



The "Area" Bound Between Two Curves function

$$A_{Area} = \int_a^b [f - g]\, dx$$

The Bound "Area" Between Two Curves along the ($\boldsymbol{x-axis}$):

$$f(x) \geq g(x)$$

$$A_{Area}(x) = \int_a^b [Upper - Lower]\, dx = \int_a^b [f(x) - g(x)]\, dx$$

The Bound "Area" Between Two Curves along the ($\boldsymbol{y-axis}$):

$$f(y) \geq g(y)$$

$$A_{Area}(y) = \int_a^b [Right - Left]\, dy = \int_a^b [f(y) - g(y)]\, dy$$

There are three (3) different types of "Area" bounded between two curves:

- **"Area" bounded "Between" Curves that "*Cross*".**
- **"Area" bounded "Between" Curves that "*Do Not Cross*".**
- **"Area" bounded "Between" Curves that are "*Bound at the Limits*".**

The Process for finding the Area between Two Curves:

1. If the limits of integration along the interval $[a, b]$, are **not** provided, it means that you are in the situation where the two curves cross and we are asking you to find the area bounded by the curves between the intersection points ("Bound at the limits).

   In this case, set the functions equal, to each other, and solve for the point of intersection.

   If the limits of integration are given, yet the problem states that the curves **do** cross, you still need to find the intersection points, as you will need to switch between the two functions and find two integrals (more details to follow in the examples).

2. Select a **test point** between the interval $[a, b]$, to determine which of the two functions has the "Larger" value, and which has the "Smaller" value.

   $$f \leq g \qquad \text{or} \qquad f \geq g$$

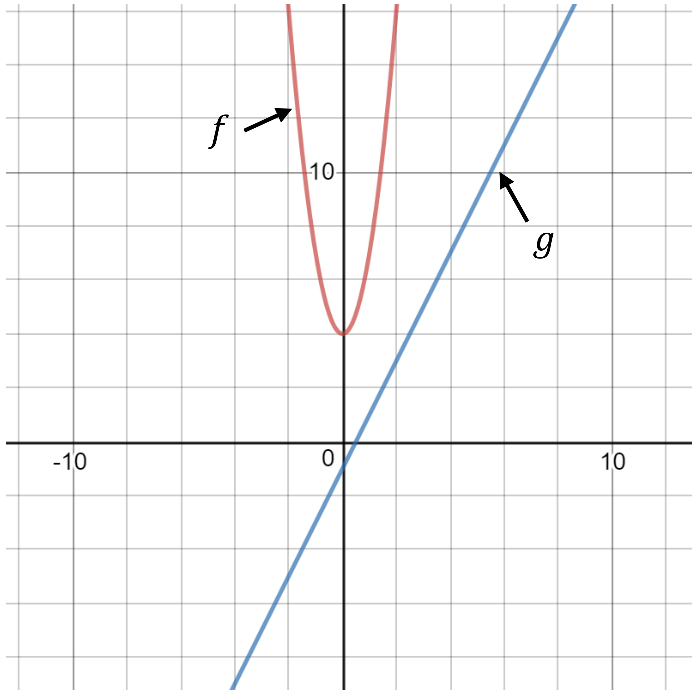
3. Integrate the two-function difference on each interval $[a, b]$:

   a "**Top** Upper" minus "**Bottom** Lower" $\rightarrow$ dx

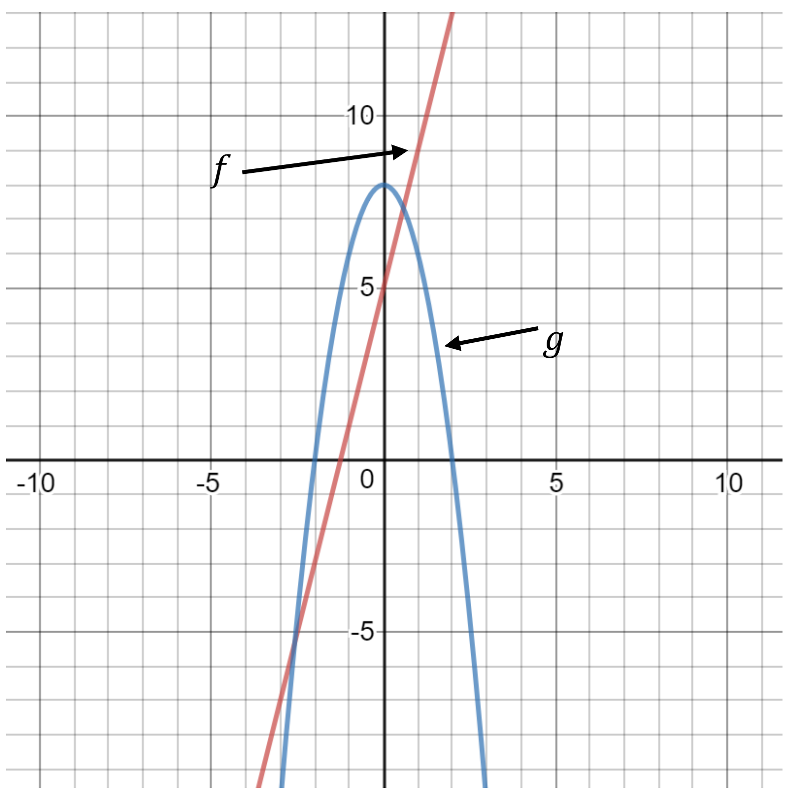
   b "**Right**" minus "**Left**" $\rightarrow$ dy

   c Function with **"Larger"** values minus Function with **"Smaller"** values over the given interval $[a, b]$.

"Area" bounded "Between" Curves that "*Do Not Cross*":



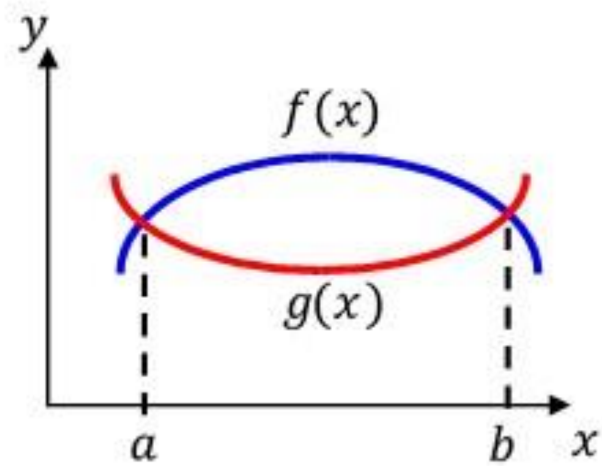
"Area" bounded "Between" Curves that "*Cross*":

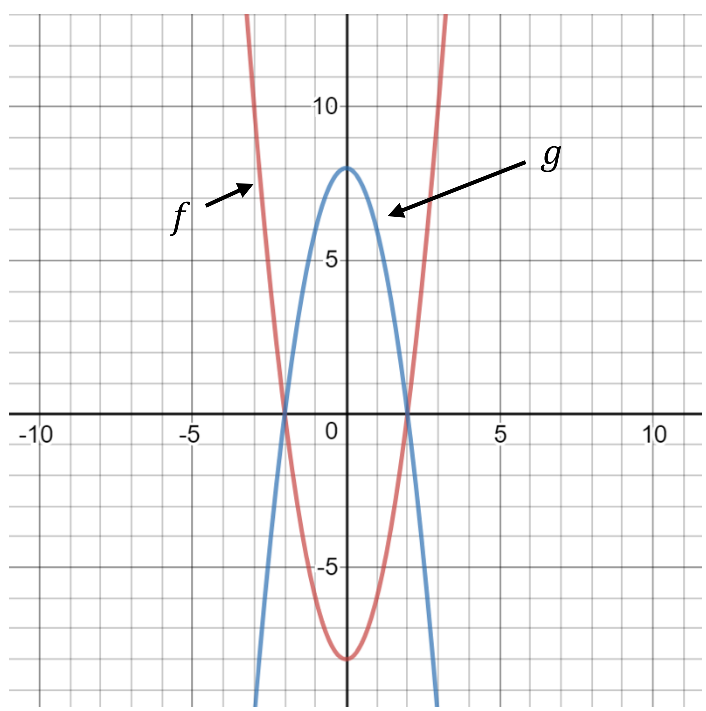


"Area" bounded "Between" Curves that are "*Bound at the Limits*":

It is sometimes useful to find the area bounded by two curves, without being told the starting and ending x-values.

In such problems, the curves completely enclose an area, and the x-values for the upper and lower limits of integration are found by setting the functions equal and solving. The x-values [**a, b**] along the domain are where the curves meet.





Function Toolkit

There are some basic functions which are helpful to know the name and shape of the graph of the function.

We call these the basic toolkit of functions.

For these definitions, we will use (x) as the input variable and ($f(x)$) as the output variable.

Functional Toolkit

The various functions ($f(x)$),

Constant: $f(x) = C$, where (C) is a constant number

Identity: $f(x) = x$

Absolute Value: $f(x) = |x|$

Quadratic: $f(x) = x^2$

Cubic: $f(x) = x^3$

Reciprocal

$$f(x) = \frac{1}{x}$$

Reciprocal Squared

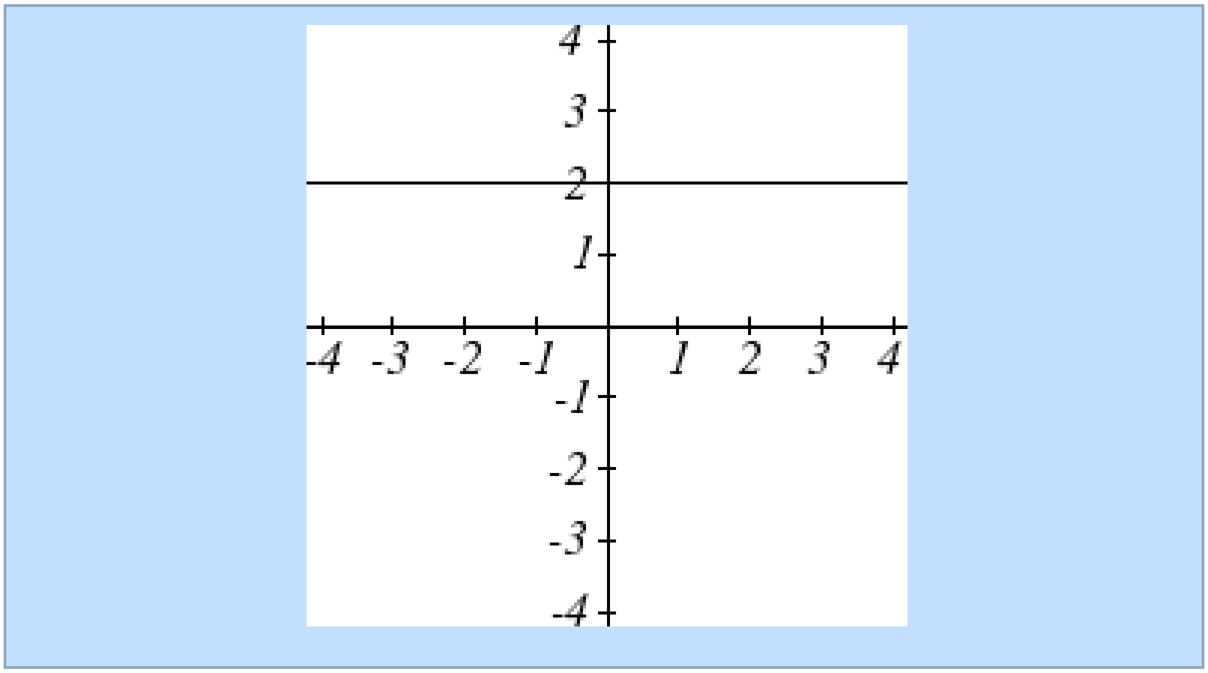
$$f(x) = \frac{1}{x^2}$$

Square Root $f(x) = \sqrt{x}$

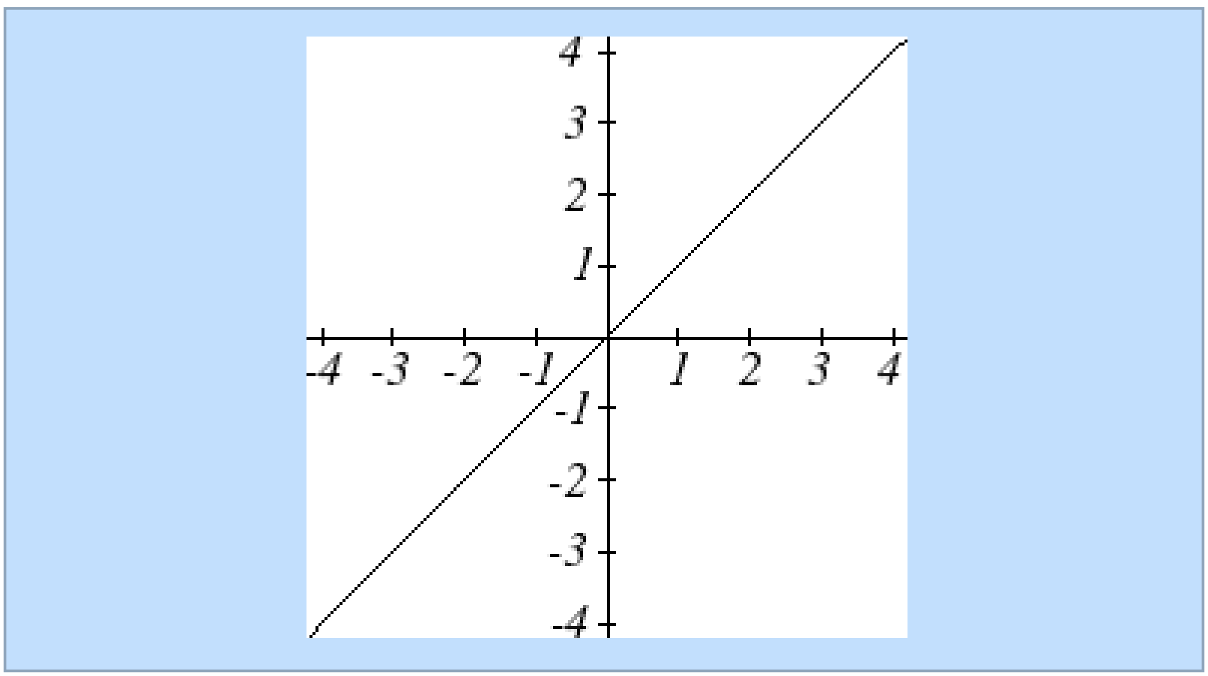
Cube Root $f(x) = \sqrt[3]{x}$

For these definitions, we will use (x) as the input variable and ($f(x)$) as the output variable.

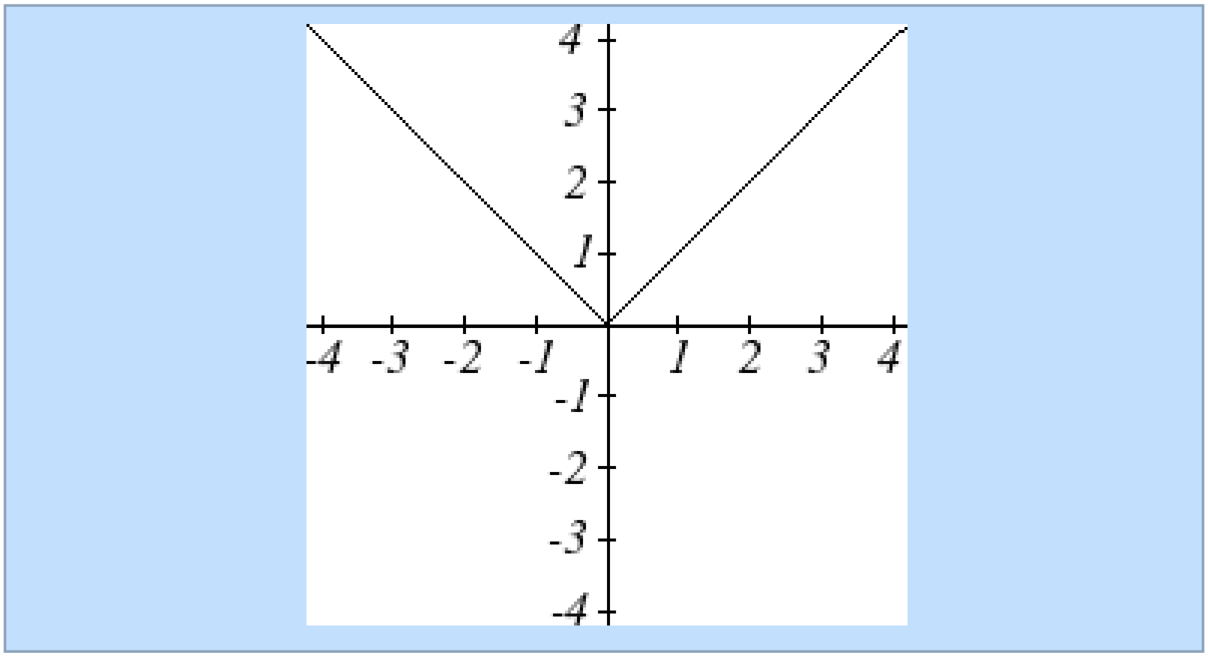
Constant: $f(x) = C$, where ($C = 2$) is a constant



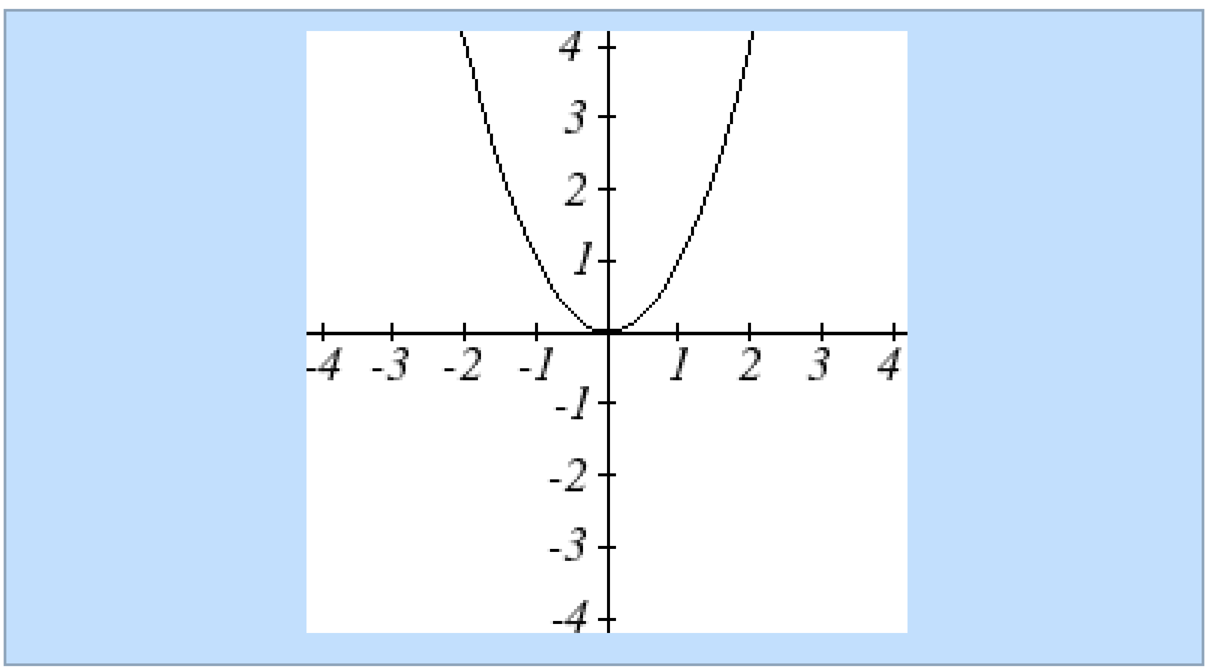
Identity: $f(x) = x$



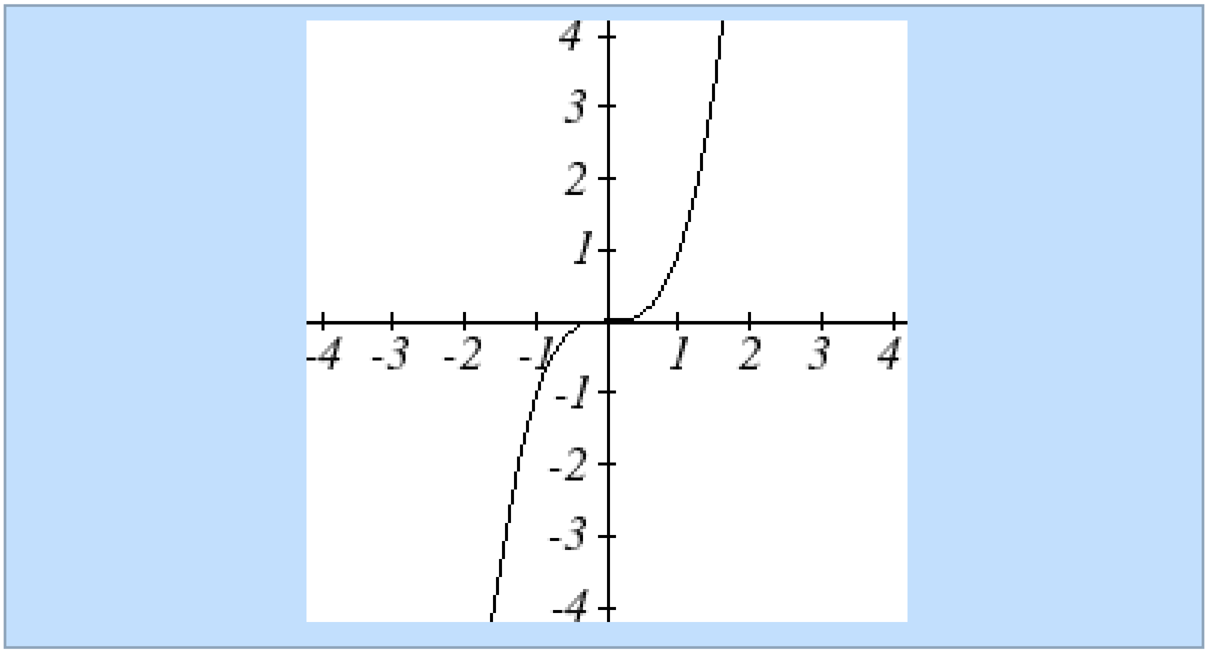
Absolute Value: $f(x) = |x|$



Quadratic: $f(x) = x^2$

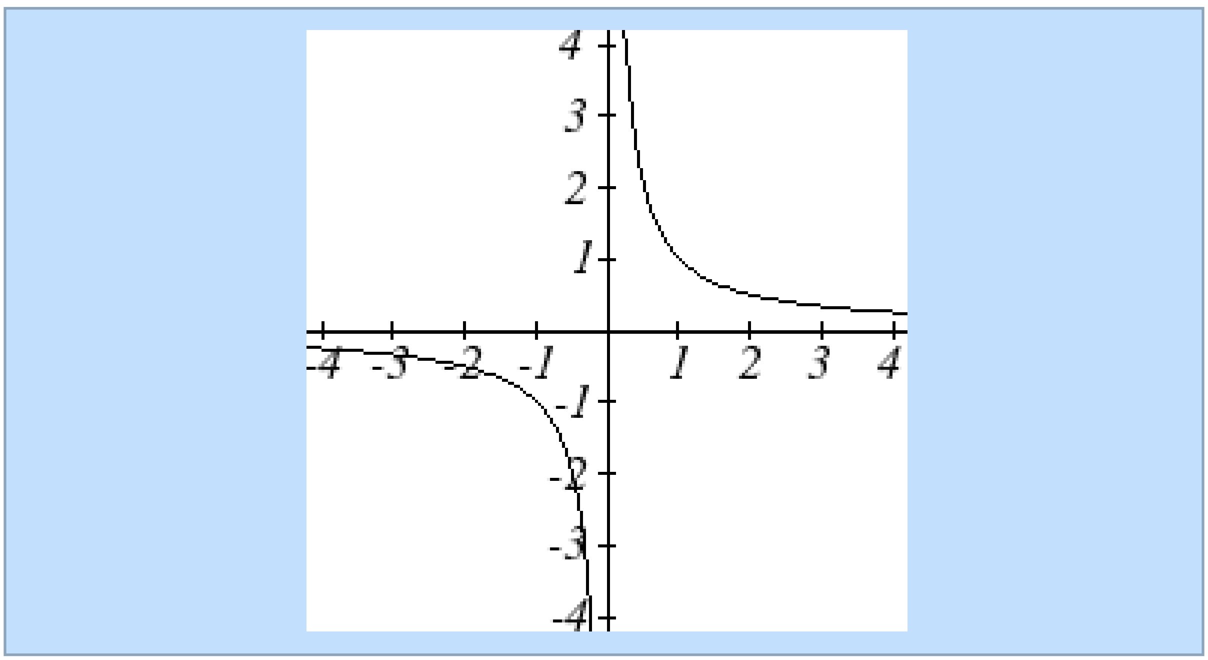


Cubic: $f(x) = x^3$



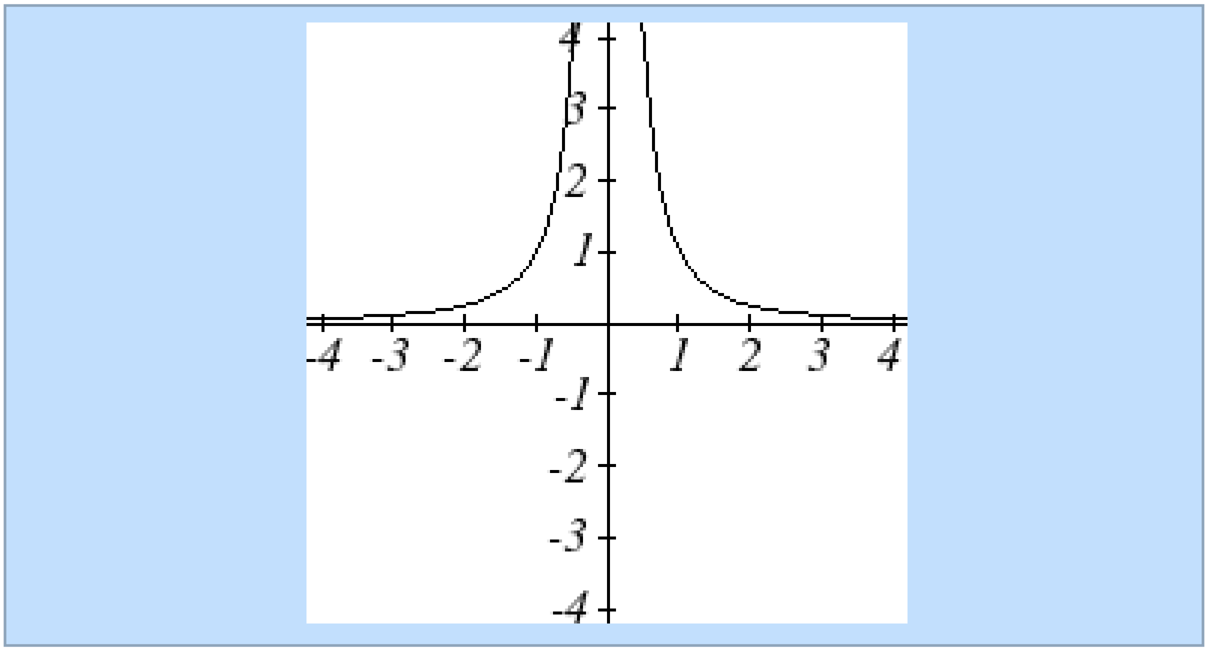
Reciprocal

$$f(x) = \frac{1}{x}$$

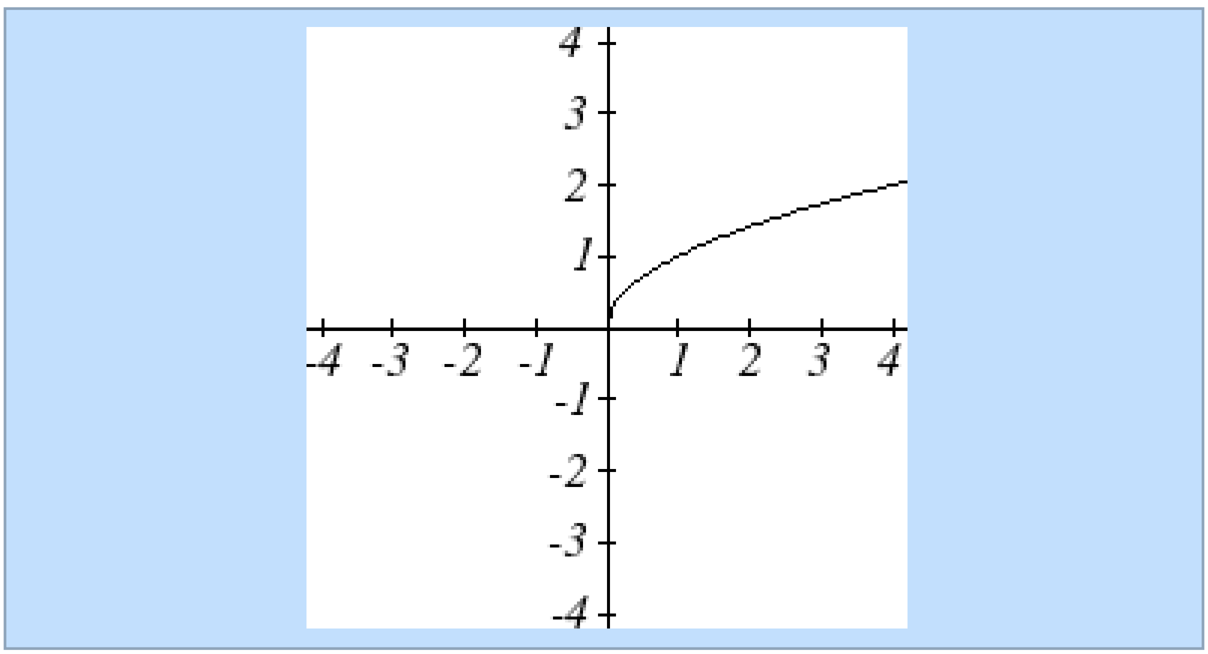


Reciprocal Squared

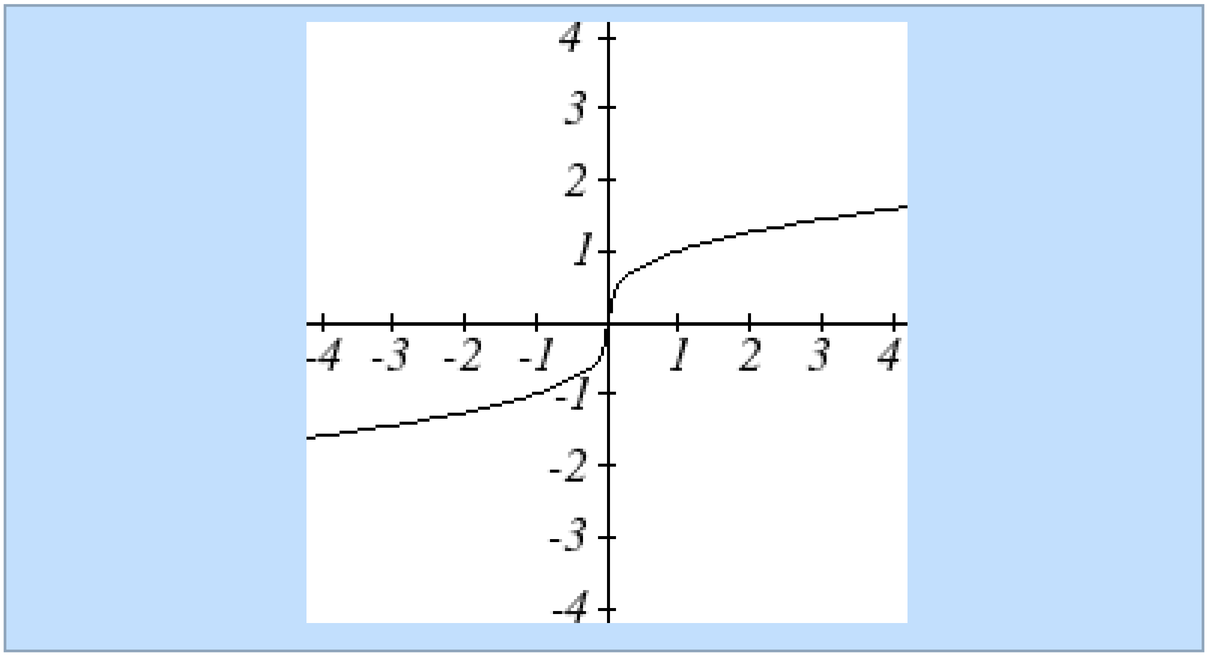
$$f(x) = \frac{1}{x^2}$$



Square Root $f(x) = \sqrt{x}$



Cube Root $f(x) = \sqrt[3]{x}$

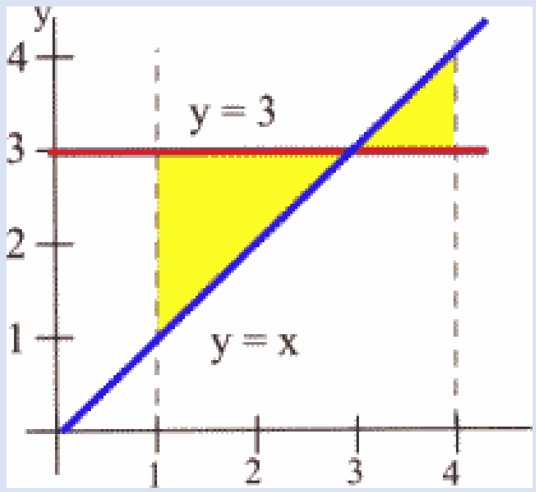


Example Problem #1

Find the **"Area" bounded "Between" Curves that "Cross"**, the graphs of:

$$f(x) = x \quad \text{and} \quad g(x) = 3$$

$$\text{for} \quad 1 \le x \le 4$$



Always start with a graph so you can see which graph is the top and which is the bottom.

In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces. Geometrically, we can see that the Total Area is:

$$A = \left(\frac{1}{2}bh\right)_1 + \left(\frac{1}{2}bh\right)_2 = 2 + 0.5 = 2.5 \ Square\ Units$$

Writing the area as a sum of definite integrals, we get.

$$A_{Area} = \int_a^b [f(x) - g(x)]\,dx$$

$$[g(x) - f(x)] = (3 - x) \quad | \quad [f(x) - g(x)] = (x - 3)$$

$$g(x) \ge f(x) \quad | \quad f(x) \ge g(x)$$

$$A_{Area} = \int_1^3 (3 - x)\,dx + \int_3^4 (x - 3)\,dx$$

Example Problem #1 – Cont'd

These integrals are easy to evaluate using antiderivatives:

$$A_{Area} = \left[3x - \frac{1}{2}x^2\right]_1^3 + \left[\frac{1}{2}x^2 - 3x\right]_3^4$$

$$A_{Area} = \left[\left(3(3) - \frac{1}{2}(3)^2\right) - \left(3(1) - \frac{1}{2}(1)^2\right)\right] + \left[\left(\frac{1}{2}(4)^2 - 3(4)\right) - \left(\frac{1}{2}(3)^2 - 3(3)\right)\right]$$

$$A_{Area} = \left[\left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right)\right] + \left[(8 - 12) - \left(\frac{9}{2} - 9\right)\right]$$

$$A_{Area} = \left[\frac{9}{2} - \frac{5}{2}\right] + \left[-4 - \left(-\frac{9}{2}\right)\right]$$

$$A_{Area} = 2 + \left[-4 + \frac{9}{2}\right] = 2 + \frac{1}{2}$$

$$A_{Area} = \frac{5}{2} \; Square\ Units = 2.5 \; Square\ Units$$

The sum of these two integrals tells us that the total area between (f) and (g) is [**2.5 square units**], which we already knew from the picture.

In the graph above the triangle on the right, the graph of $(y = x)$ is above the graph of $(y = 3)$, so the integrand $(f(x) = (3 - x))$, is negative; in the definite integral, the area of that triangle comes in with a negative sign.

In this example, it was easy to see exactly where the two curves crossed so we could break the region into the two pieces to figure separately.

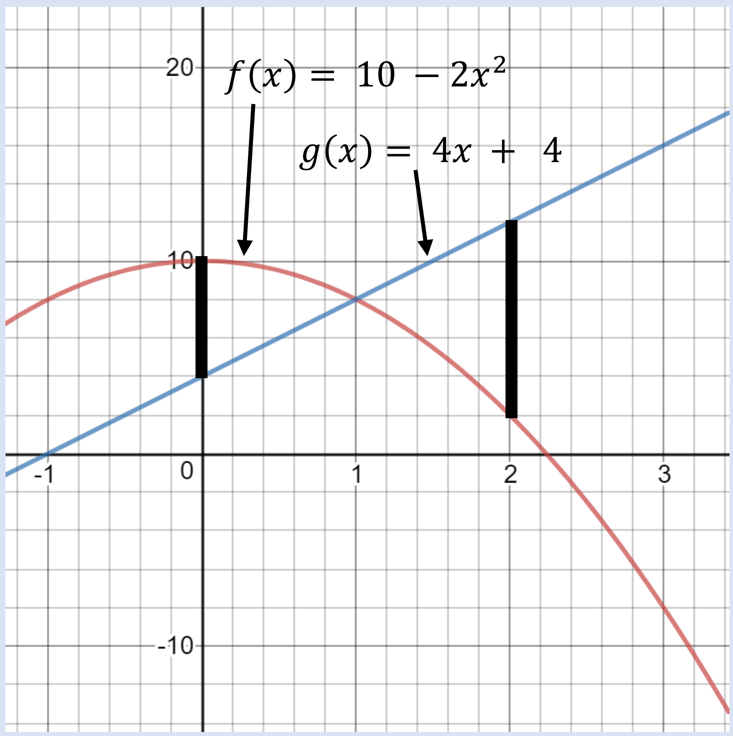
In other examples, you might need to solve an equation to find where the curves cross. At the point where the curves cross, the "Upper" curve becomes the "Lower" curve, and the "Lower" curve, becomes the "Upper" curve.

Example Problem #2

Find the **"Area" bounded "Between" Curves that "Cross"**, the graphs of:

$$y_1 = f(x) = 10 - 2x^2 \quad \text{and} \quad y_2 = g(x) = 4x + 4$$

$$\text{for} \quad 0 \le x \le 2$$



Solution:

Always start with a graph so you can see which graph is the top and which is the bottom.

In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces.

To find the intersection point, we set the functions equal and solve.

$$10 - 2x^2 = 4x + 4$$

$$2x^2 + 4x - 6 = 0$$

$$(2x - 2)(x + 3) = 0$$

Example Problem #2 - Cont'd

Where the solutions are,

$$x = 1 \qquad \& \qquad x = -3$$

The solution ($x = -3$) is not in the interval [0, 2],

The curves cross at the interval ($x = 1$), so we must integrate separately over the interval [0, 1] and [1, 2]. The Critical Limits (CL) are the following:

$$CL = \begin{cases} x = 0 \\ x = 1 \\ x = 2 \end{cases}$$

Writing the area as a sum of definite integrals, we get.

$$A_{Area} = \int_a^b [f(x) - g(x)]\, dx$$

$$A_{Area} = \int_0^1 [y_1 - y_2]\, dx + \int_1^2 [y_2 - y_1]\, dx$$

$$A_{Area} = \int_0^1 \begin{bmatrix} (10 - 2x^2) \\ - (4x + 4) \end{bmatrix} dx + \int_1^2 \begin{bmatrix} (4x + 4) \\ - (10 - 2x^2) \end{bmatrix} dx$$

$$A_{Area} = \int_0^1 [-2x^2 - 4x + 6]\, dx + \int_1^2 [2x^2 + 4x - 6]\, dx$$

$$A_{Area} = \left[-\frac{2}{3}x^3 - 2x^2 + 6x\right]_0^1 + \left[\frac{2}{3}x^3 + 2x^2 - 6x\right]_1^2$$

$$A_{Area} = \left[-\frac{2}{3} - 2 + 6\right] + \left[\left(\frac{16}{3} + 8 - 12\right) - \left(\frac{2}{3} + 2 - 6\right)\right]$$

$$A_{Area} = \left[\frac{10}{3} + \frac{4}{3} - \frac{10}{3}\right] = 1.33 \;\; Square\ Units$$

Therefore, the "Area" between two curves that cross is [**1.33 Square units**].

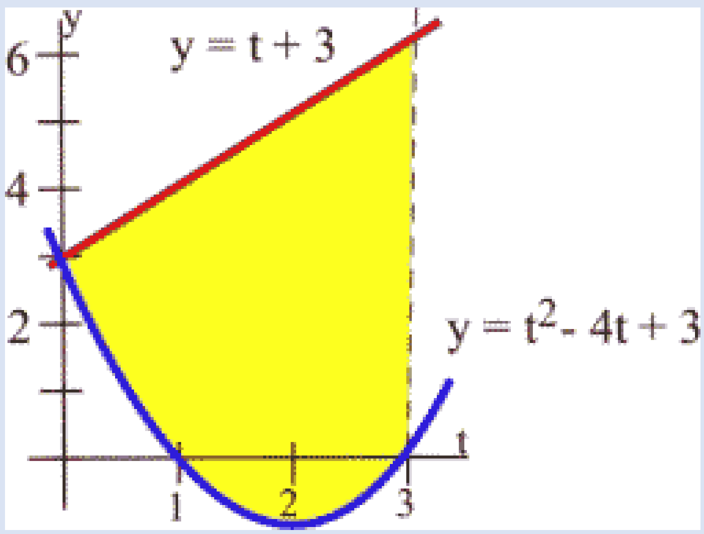
Example Problem #3

Find the **"Area" bounded "Between" Curves that "Do Not Cross"**

Two objects start from the same location and travel along the same path with velocities:

$$v_A(t) = (t + 3)\, m/s \quad \text{and} \quad v_B(t) = (t^2 - 4t + 3)\, m/s$$

How far ahead is (A) after (3 seconds)?



Solution:

1. Always start with a graph so you can see which function is "Upper" and "Lower".

In this example, the two curves cross, in only one place.

To find the intersection point, we set the functions equal and solve.

$$t + 3 = t^2 - 4t + 3$$

$$t^2 - 5t = 0$$

$$t \cdot (t - 5) = 0$$

Example Problem #3 – Cont'd

The Critical Limits (CL) are the following:

$$CL = \begin{cases} t = 0 \\ t = 5 \end{cases}$$

However, we are looking or the time after ($t = 3$ seconds); thus:

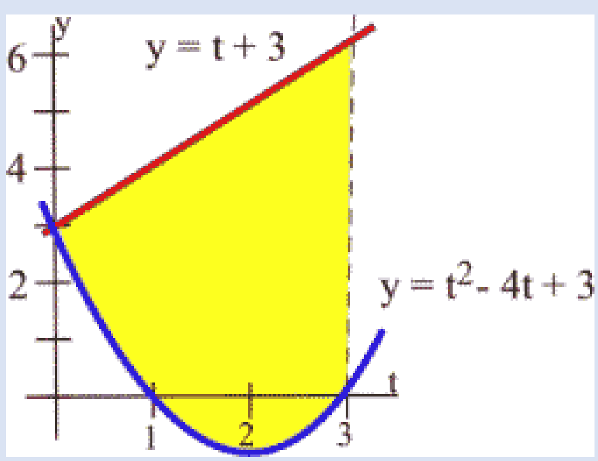
The Critical Limits (CL) are the following:

$$CL = \begin{cases} t = 0 \\ t = 3 \end{cases}$$

2. Next, let's graph so you can see which function is "Upper" and "Lower"

3. Also, let's create at test point between ($0 \leq t \leq 3$), to get a number that determines which function is the "Upper" and "Lower"

t	$v_A(t) = (t + 3)$	$v_B(t) = (t^2 - 4t + 3)$	Notes
0	3	3	"Lower" Limit
1	4	0	$v_A(t) \geq v_B(t)$
3	6	0	Restricted Limit
5	8	8	"Upper" Limit



Example Problem #3 – Cont'd

Therefore, according, to the table and the plot:

$$Upper \rightarrow \quad v_A(t) \quad \geq \quad v_B(t) \quad \leftarrow Lower$$

$$Upper \rightarrow \quad (t+3) \quad \geq \quad (t^2 - 4t + 3) \quad \leftarrow Lower$$

The area between the graphs of ($v_A(t)$) and ($v_B(t)$), represents the distance between the objects, after (3 seconds).

Writing the area as a sum of definite integrals, we get.

$$s(t) = \int_a^b [v_A(t) - v_B(t)]\, dt$$

$$s(t) = \int_0^3 [(t + 3) - (t^2 - 4t + 3)]\, dt$$

$$s(t) = \int_0^3 [-t^2 + 5t]\, dt$$

$$s(t) = \left[-\frac{1}{3}t^3 + \frac{5}{2}t^2\right]_0^3$$

$$s(t) = \left[\left(-\frac{1}{3}(3)^3 + \frac{5}{2}(3)^2\right) - 0\right]$$

$$s(t) = \left[-9 + \frac{45}{2}\right] = \left[\frac{-18 + 45}{2}\right] = \frac{27}{2}$$

$$s(t) = \frac{27}{2}\ meters = 13.5\ meters$$

Example Problem #4

Find the **"Area" bounded "Between" Curves that are "*Bound at the Limits*"**

Find/Sketch the Enclosed Area between the given curves:

$$y = \sqrt{x - 1} \qquad \text{and} \qquad x - y = 1$$

Solution:

1. Always start with a graph so you can see which function is "Upper" and "Lower".

In this example, the two curves cross (intersect), in two places.

2. Decide which direction you want to graph and integrate.

Integrating along ($x - axis$)	Integrating along ($y - axis$)
$y_1 = \sqrt{x - 1}$	$x_1 = y^2 + 1$
$y_2 = x - 1$	$x_2 = y + 1$

For this example, we will choose to integrate along the ($\boldsymbol{x - axis}$)

To find the intersection points, we set the functions equal and solve.

$$\sqrt{x - 1} = x - 1$$

$$x - 1 = (x - 1)^2$$

$$x - 1 = x^2 - 2x + 1$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

The Critical Limits (CL) are the following:

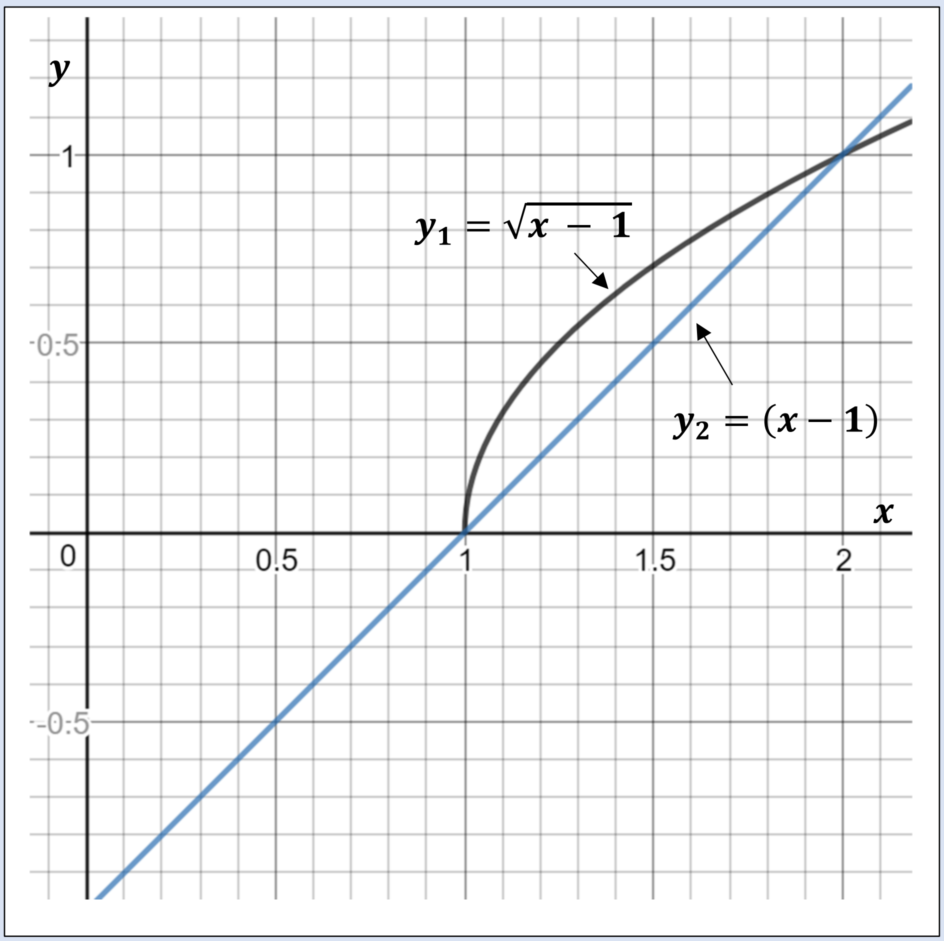
$$CL = \begin{cases} x = 1 \\ x = 2 \end{cases}$$

Example Problem #4 – Cont'd

3. Next, let's graph so you can see which function is "Upper" and "Lower"

4. Also, let's create a test point between ($1 \leq x \leq 2$), to get a number that determines which function is the "Upper" and "Lower"

x	$y_1 = \sqrt{x-1}$	$y_2 = (x-1)$	Notes
1	0	0	"Lower" Limit
$\frac{3}{2}$	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{1}{2} = 0.5$	$y_1 \geq y_2$
2	1	1	"Upper" Limit



Example Problem #4 – Cont'd

Therefore, according, to the table and the plot:

$$Upper \rightarrow \quad y_1 \;\geq\; y_2 \quad \leftarrow Lower$$

$$Upper \rightarrow \quad \sqrt{x - 1} \;\geq\; (x - 1) \quad \leftarrow Lower$$

Writing the area ($A(x)$) as a sum of definite integrals, we get.

$$A(x) = \int_1^2 [y_1(x) - y_2(x)]\, dx$$

$$A(x) = \int_1^2 \left[(x - 1)^{\frac{1}{2}} - (x - 1)\right] dx$$

$$A(x) = \int_1^2 (x - 1)^{\frac{1}{2}}\, dx - \int_1^2 (x - 1)\, dx$$

$$A(x) = \frac{2}{3}\left[(x - 1)^{\frac{3}{2}}\right]_1^2 - \left[\frac{1}{2}x^2 - x\right]_1^2$$

$$A(x) = \frac{2}{3}\left[((2) - 1)^{\frac{3}{2}} - ((1) - 1)^{\frac{3}{2}}\right] - \left[\left(\frac{1}{2}(2)^2 - (2)\right) - \left(\frac{1}{2}(1)^2 - (1)\right)\right]$$

$$A(x) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \; squre\ units$$

Example Problem #5

Find the **"Area" bounded "Between" Curves that are "*Bound at the Limits*"**

Find/Sketch the Enclosed Area between the given curves along the $(\boldsymbol{y - axis})$:

$$y = \sqrt{x - 1} \qquad \text{and} \qquad x - y = 1$$

Solution: Using Example #4.

1. Always start with a graph so you can see which function is "Upper" and "Lower".

In this example, the two curves cross (intersect), in two places.

2. For this example, we will now choose to integrate along the $(\boldsymbol{y - axis})$.

Integrating along $(\boldsymbol{y - axis})$
$x_1 = y^2 + 1$
$x_2 = y + 1$

To find the intersection points, we set the functions equal and solve.

$$y^2 + 1 = y + 1$$

$$y^2 - y = 0$$

$$y \cdot (y - 1) = 0$$

The Critical Limits (CL) are the following:

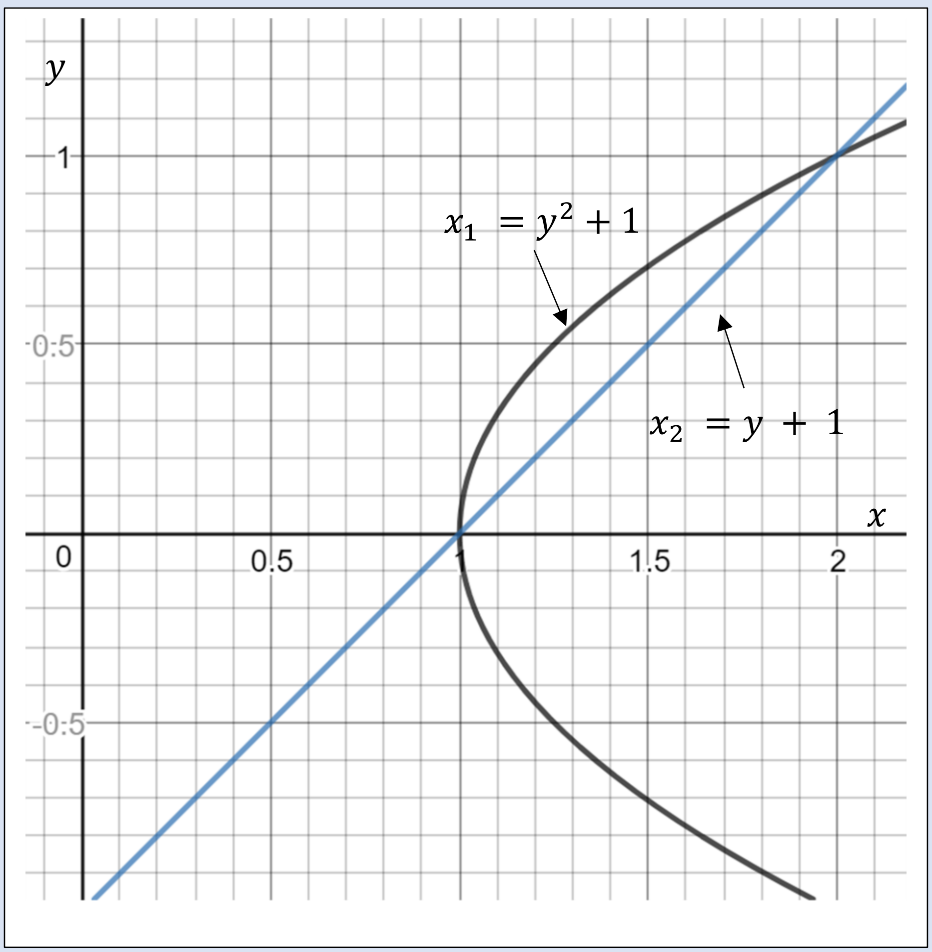
$$CL = \begin{cases} y = 0 \\ y = 1 \end{cases}$$

Example Problem #5 – Cont'd

3. Next, let's graph so you can see which function is "Right" and "Left"

4. Also, let's create at test point between ($0 \leq y \leq 1$), to get a number that determines which function is the "Right" and "Left"

y	$x_1 = y^2 + 1$	$x_2 = y + 1$	Notes
0	1	1	"Lower Limit"
$\frac{1}{2}$	$\frac{5}{4} = 1.25$	$\frac{3}{2} = 1.5$	$x_2 \geq x_1$
1	2	2	"Upper Limit"



Example Problem #5 – Cont'd

Therefore, according, to the table and the plot:

$$Right \rightarrow \quad x_2 \;\geq\; x_1 \quad \leftarrow Left$$

$$Right \rightarrow \quad (y + 1) \;\geq\; (y^2 + 1) \quad \leftarrow Left$$

Writing the area ($A(y)$) as a sum of definite integrals, we get.

$$A(y) = \int_0^1 [x_2(y) - x_1(y)]\, dy$$

$$A(y) = \int_0^1 [(y + 1) - (y^2 + 1)]dy$$

$$A(y) = \int_0^1 [y - y^2]dy$$

$$A(y) = \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$A(y) = \left[\frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right] - \left[\frac{1}{2}(0)^2 - \frac{1}{3}(0)^3\right]$$

$$A(y) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \ squre\ units$$

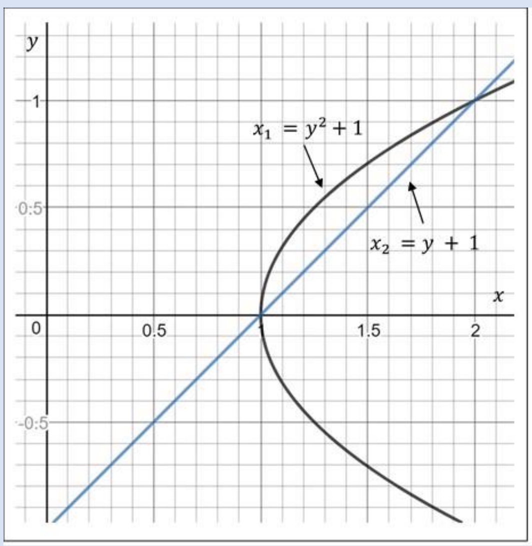
Example Problem #5 – Cont'd

Therefore, the above Enclosed Area problem, is the same whether you integrating along the ($\boldsymbol{x-axis}$), or along the ($\boldsymbol{y-axis}$).

Integrating along the ($\boldsymbol{y-axis}$)

$$f(y) = [\text{"}Right\text{"} - \text{"}Left\text{"}]$$

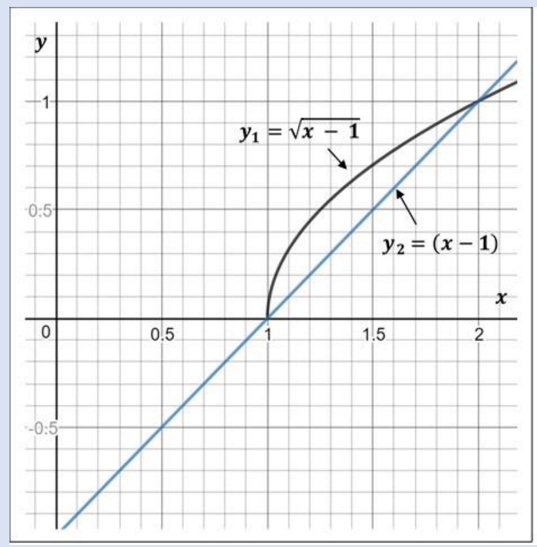
$$f(y) = [y - y^2]$$



Integrating along the ($\boldsymbol{x-axis}$)

$$f(x) = [\text{"}Upper\text{"} - \text{"}Lower\text{"}]$$

$$f(x) = \left[(x-1)^{\frac{1}{2}} - (x-1)\right]$$



Remember, when you are integrating the function ($f(y)$) along the ($\boldsymbol{y-axis}$), the Enclosed Area is equal to the "Right" minus the "Left"

And, when you are integrating the function ($f(x)$) along the ($\boldsymbol{x-axis}$), the Enclosed Area is equal to the "Upper" minus the "Lower"

Example Problem #6

Find the **"Area" bounded "Between" Curves that are "*Bound at the Limits*"**

Find/Sketch the Enclosed Area between the given curves:

$$x = y^4 \quad ; \quad y = \sqrt{2 - x} \quad ; \quad y = 0$$

Solution:

1. Always start with a graph so you can see which function is "Upper" and "Lower".

In this example, the two curves cross (intersect), in two places.

2. Decide which direction you want to graph and integrate.

Integrating along ($\boldsymbol{x - axis}$)	**Integrating along ($\boldsymbol{y - axis}$)**
$y_1 = x^{\frac{1}{4}}$	$x_1 = y^4$
$y_2 = \sqrt{2 - x}$	$x_2 = 2 - y^2$

For this example, we will choose to integrate along the ($\boldsymbol{y - axis}$)

We choose the ($\boldsymbol{y - axis}$), because ($y = 0$), is given.

To find the intersection points, we set the functions equal and solve.

$$y^4 = 2 - y^2$$

$$y^4 + y^2 - 2 = 0$$

$$(y^2 + 2)(y^2 - 1) = 0$$

The Critical Limits (CL) are the following:

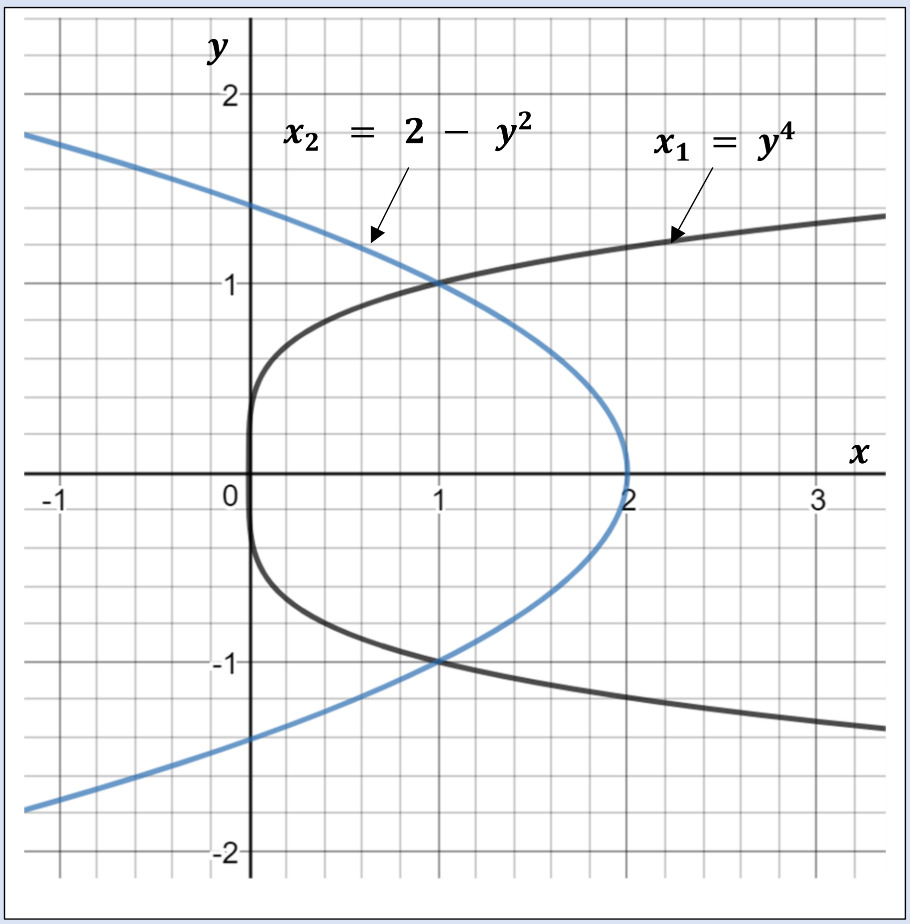
$$CL = \begin{cases} y = +1 \\ y = -1 \end{cases}$$

Example Problem #6 – Cont'd

3. Next, let's graph so you can see which function is "Right" and "Left"

4. Also, let's create at test point between ($-1 \le x \le 1$), to get a number that determines which function is the "Right" and "Left"

y	$x_1 = y^4$	$x_2 = 2 - y^2$	Notes
-1	1	1	"Lower" Limit
0	0	2	$x_2 \ge x_1$
1	1	1	"Upper" Limit



Example Problem #6 – Cont'd

Therefore, according, to the table and the plot:

$$Right \rightarrow \quad x_2 \geq x_1 \quad \leftarrow Left$$

$$Right \rightarrow \quad \sqrt{x - 1} \quad \geq \quad (x - 1) \quad \leftarrow Left$$

Writing the area ($A(y)$) as a sum of definite integrals, we get.

$$A(y) = \int_{-1}^{1} [x_2(y) - x_1(y)]\, dy$$

$$A(y) = \int_{-1}^{1} [2 - y^2 - y^4]\, dy$$

Since ($f(y) = [2 - y^2 - y^4]$), is an Even Function ($f(-y) = f(y)$); then by the law of Symmetry, we can write.

$$A(y) = 2\int_{0}^{1} [2 - y^2 - y^4]\, dy$$

$$A(y) = 2\left[2y - \frac{1}{3}y^3 - \frac{1}{5}y^5\right]_0^1$$

$$A(y) = 2\left[\left(2(1) - \frac{1}{3}(1)^3 - \frac{1}{5}(1)^5\right) - (0 - 0 - 0)\right]_0^1$$

$$A(y) = 2\left(2 - \frac{1}{3} - \frac{1}{5}\right) = 2\left(\frac{5}{3} - \frac{1}{5}\right) = 2\left(\frac{22}{15}\right)$$

The Total Area ($A(y)$) is given by:

$$A(y) = 2 \cdot \left(\frac{22}{15}\right) = \frac{44}{15}$$

The Top or Bottom half of the Area is given by:

$$\frac{1}{2}A(y) = \left(\frac{22}{15}\right)$$

Example Problem #7

Find the **"Area" bounded "Between" Curves that are "*Bound at the Limits*"**

Find/Sketch the Enclosed Area between the given curves along the $(\boldsymbol{y - axis})$:

$$x = y^4 \quad ; \quad y = \sqrt{2 - x} \quad ; \quad y = 0$$

Solution: Using Example #6.

1. Always start with a graph so you can see which function is "Upper" and "Lower". This is done by setting the equations equal to each.

In this example, the two curves cross (intersect), in two places.

2. For this example, we will choose now to integrate along the $(\boldsymbol{x - axis})$.

Integrating along $(\boldsymbol{x - axis})$
$y_1 = x^{\frac{1}{4}}$
$y_2 = \sqrt{2 - x}$

To find the intersection points, we set the functions equal and solve.

$$x^{\frac{1}{4}} = \sqrt{2 - x}$$

$$x = (2 - x)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

The Critical Limits (CL) are the following:

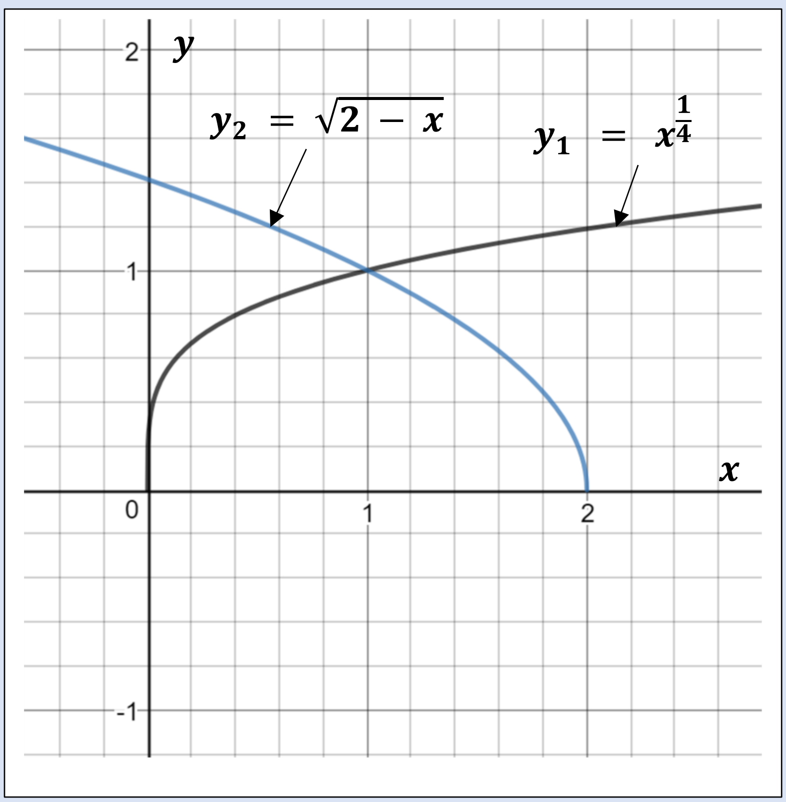
$$CL = \begin{cases} x = 1 \\ x = 4 \end{cases}$$

Example Problem #7 – Cont'd

3. Next, let's graph so you can see which function is "Upper" and "Lower"

4. Also, let's create at test point between ($1 \leq x \leq 4$), to get a number that determines which function is the "Upper" and "Lower"

x	$y_1 = x^{\frac{1}{4}}$	$y_2 = \sqrt{2 - x}$	Notes
1	1	1	$y_1 = y_2$
2	$\sqrt[4]{2} = 1.189$	0	$y_1 \geq y_2$
3	$\sqrt[4]{3} = 1.131$	$\sqrt{-1}$	Does not Exist
4	$\sqrt{2} = 1.414$	$\sqrt{-2}$	Does not Exist



Example Problem #7 – Cont'd

Therefore, the above problem, integrating along the ($\boldsymbol{x-axis}$), does not produce an enclosed Area.

The "Upper" minus "Lower" Enclosed Area is undefined:

$$f(x) = \text{"}Upper\text{"} - \text{"}Lower\text{"} = Undefined$$

Writing the area ($A(y)$) as a sum of definite integrals, we get.

$$A(x) = \int_1^4 f(x)\,dx = Undefined\,Area$$

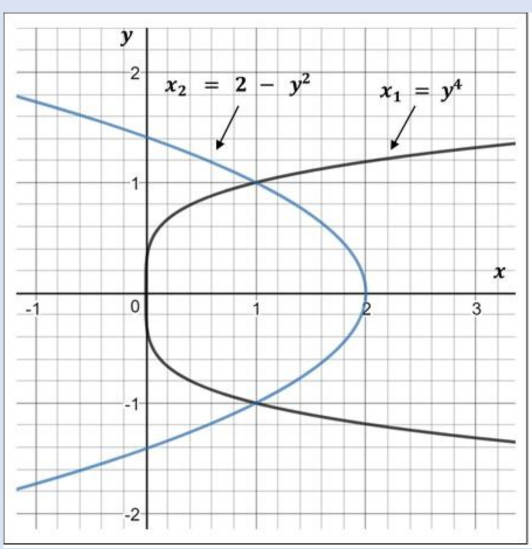
There are no official bound limits for this case, integrating along the ($\boldsymbol{x-axis}$).

However, as demonstrated in Example #6, there are bound limits when you, integrating along the ($\boldsymbol{y-axis}$).

Integrating along the ($\boldsymbol{y-axis}$)

$$f(y) = [\text{"}Right\text{"} - \text{"}Left\text{"}]$$

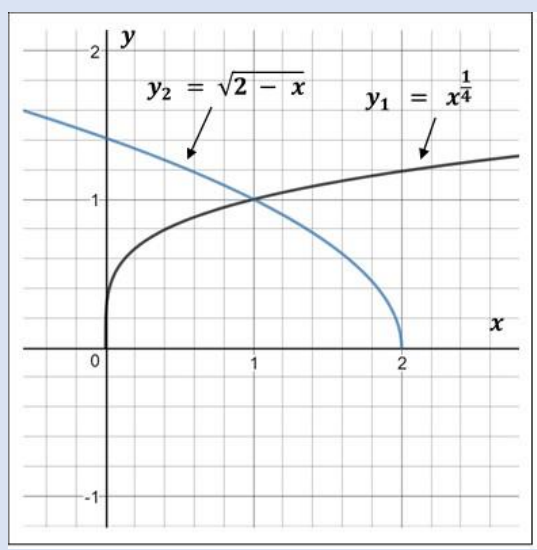
$$f(y) = [2 - y^2 - y^4]$$



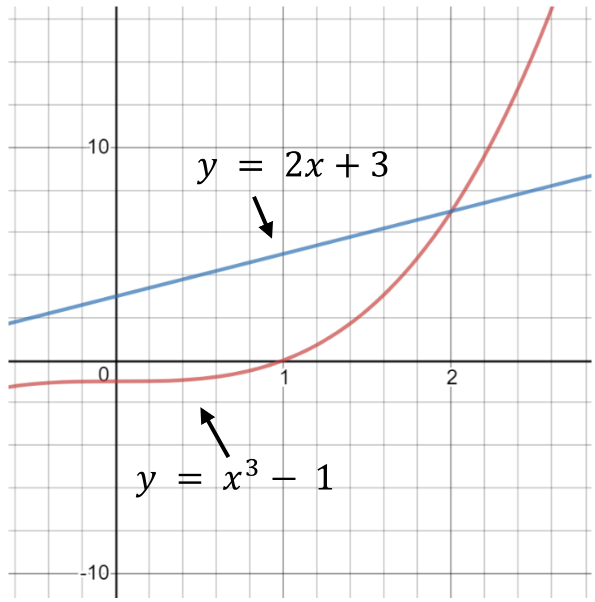
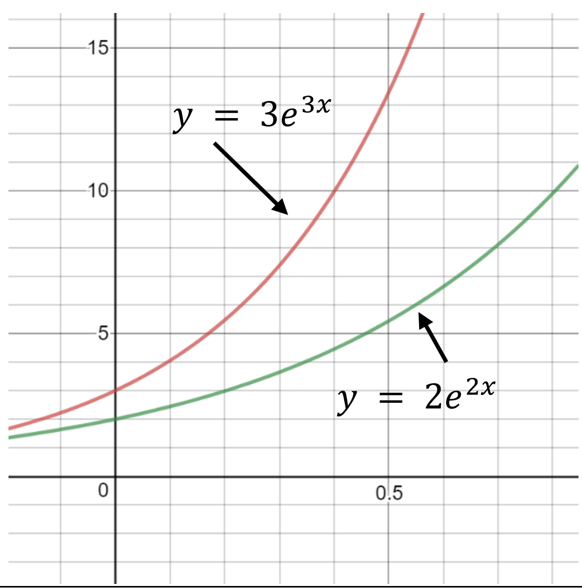
Integrating along the ($\boldsymbol{x-axis}$)

$$f(x) = [\text{"}Upper\text{"} - \text{"}Lower\text{"}]$$

$$f(x) = Undefined$$



5.6 - EXERCISES

	Find the Area between the two curves.
1.	Find the area between the curve ($y = x^3 - 1$) and the line ($y = 2x + 3$) (shown below) from [$x = 0$ to $x = 2$]. 
2.	Find the area between the curve ($y = 3e^{3x}$) and the curve ($y = 2e^{2x}$) (shown below) from [$x = 0$ to $x = 0.5$]. 

Sketch each parabola and line on the same graph and find the area between them from $[x = 0 \quad to \quad x = 2]$.			
3.	$y = x^2 + 9$ and $y = 4x + 1$	4.	$y = 2x^2 + 1$ and $y = 3x + 4$

Find the Area bounded by the given Curves.	
5.	$y = 2 - 3x^2$ and $y = x^2 - 2$
6.	$y = -2x^2 + x + 2$ and $y = 3x^2 - 2$
7.	$y = 3x^2 - 5x$ and $y = 5x$

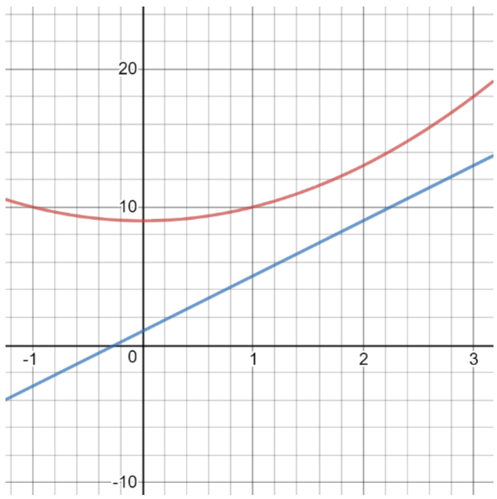
Find the Area between Curves that Cross.	
8.	$y = 6x - 3$ and $y = -6x^2 + 8x + 1$ From x = 0.5 to x = 2
9.	$y = 5x$ and $y = -12x^2 + 3x + 4$ From x = 0 to x = 1
10.	$y = 9x$ and $y = 4x^3$ From x = 1 to x = 2
11.	A Crypto-Currency Business is making profits buying and selling Bitcoin, at its website, at the rate of ($3e^{0.3t}$) million dollars per year. The company plans to invest in a new website and trading software to attract new clients and predicts profits will grow to ($8e^{0.1t}$) million dollars per year, where (t) is the number of years from now. Find the extra profits that results from the new investments of its website during the first three years ($t = 0$) to ($t = 3$).

Solutions

54. $8 \ Square\ Units$

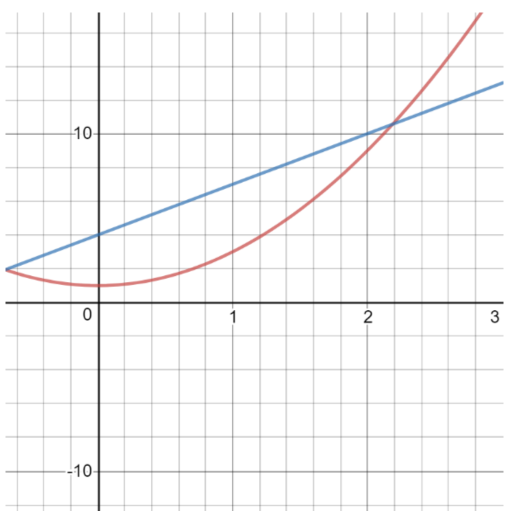
55. $e^{\frac{3}{2}} - e = 1.76 \ Square\ Units$

56. a.



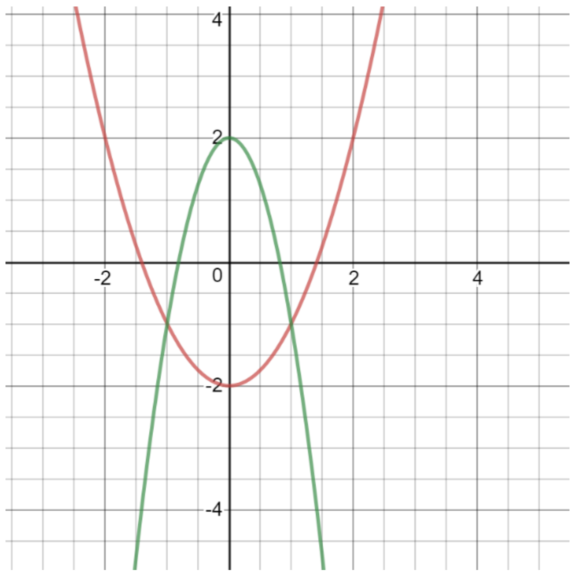
b. $\frac{32}{3} = 10.66 \ Square\ Units$

57. a.

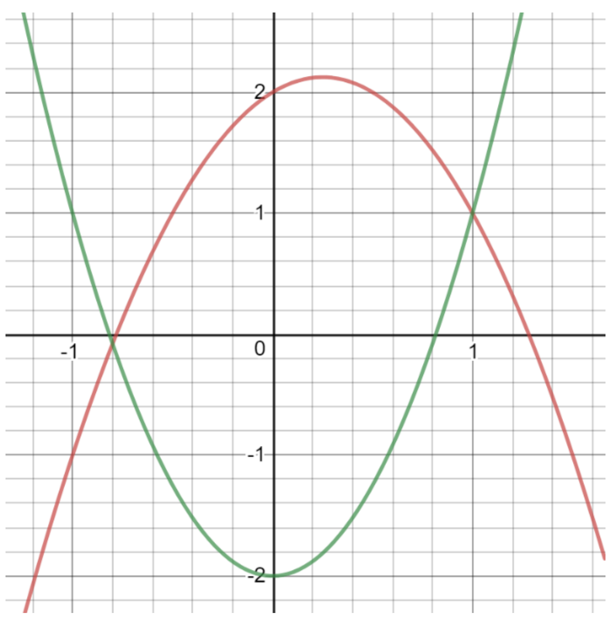


b. $\frac{20}{3} = 6.6666 \quad Square\ Units$

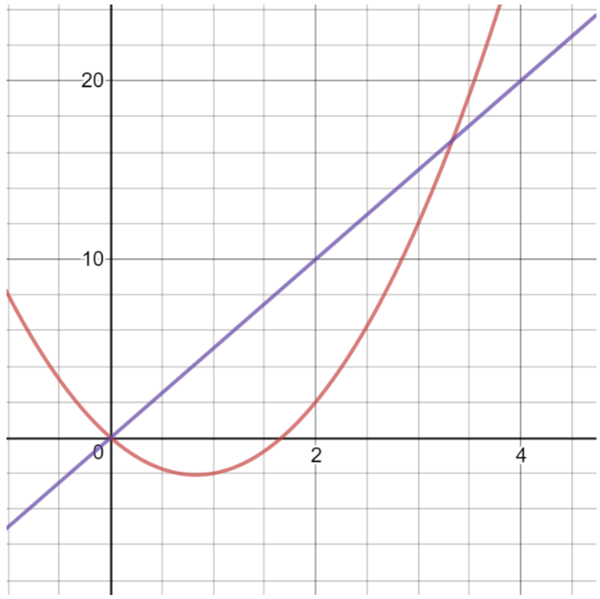
58. $\frac{16}{3} = 5.33\ Square\ Units$



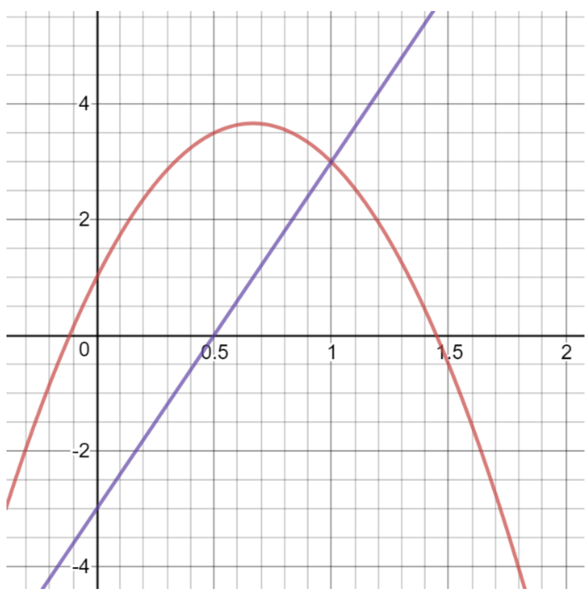
59. $4.86\ Square\ Units$



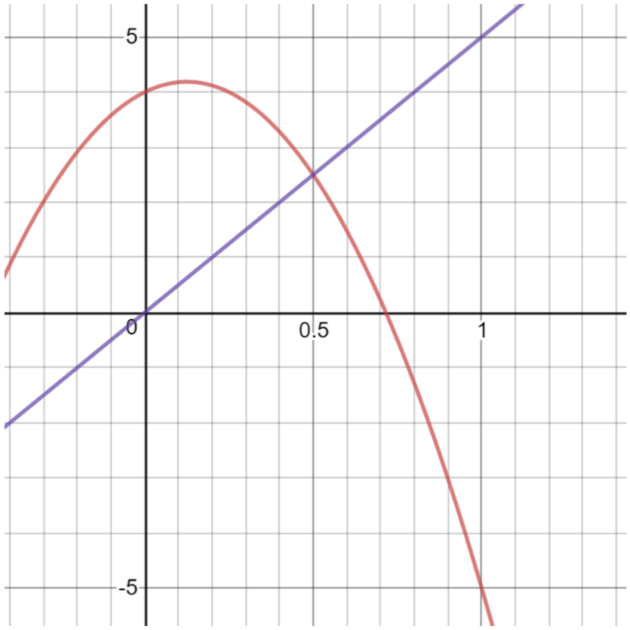
60. $18.518\ \mathit{Square\ Units}$



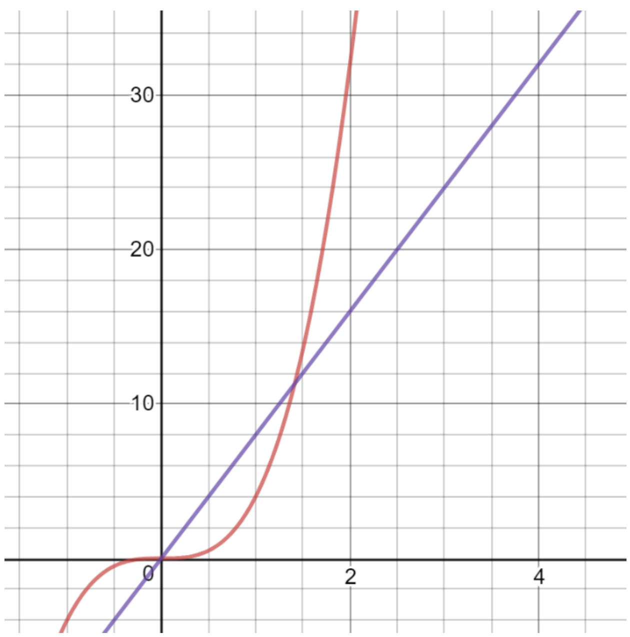
61. $8\ \mathit{Square\ Units}$



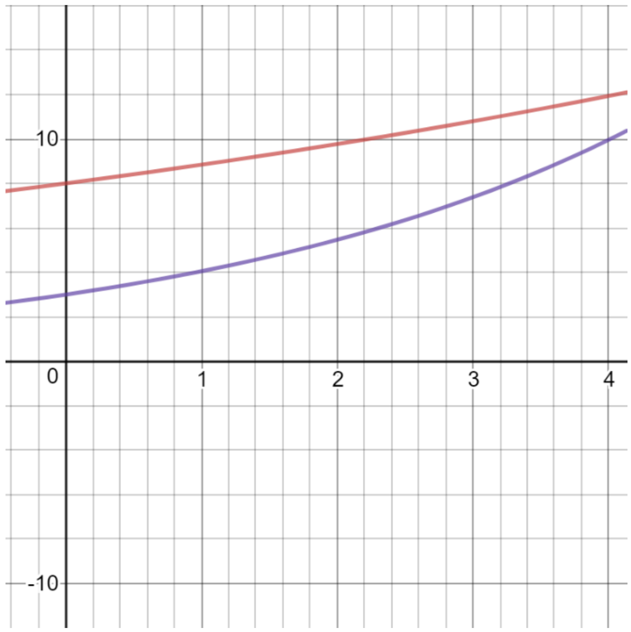
62. $3.5\ Square\ Units$

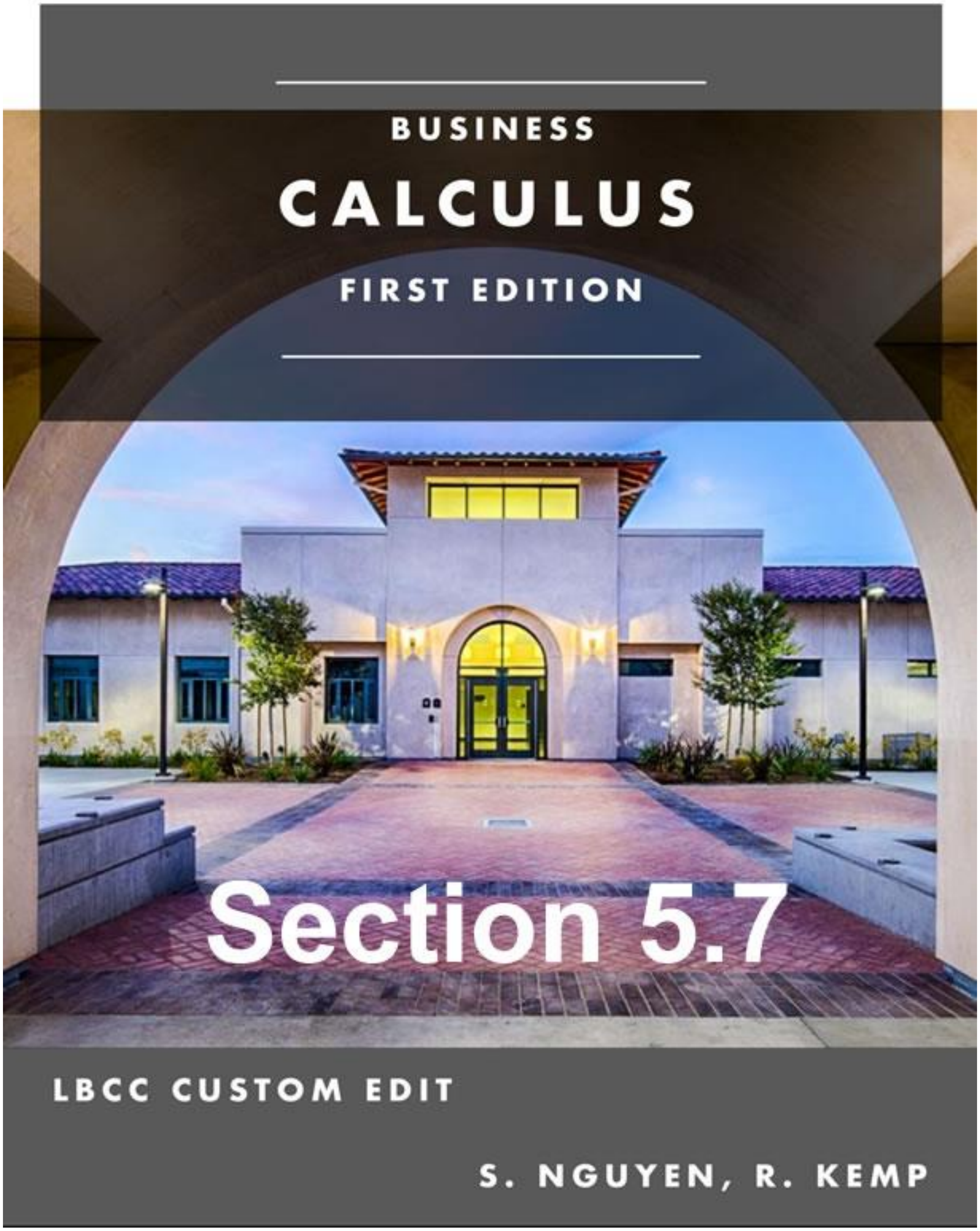


63. $4.625\ Square\ Units$



64. $About\ \ 13.4\ Million$

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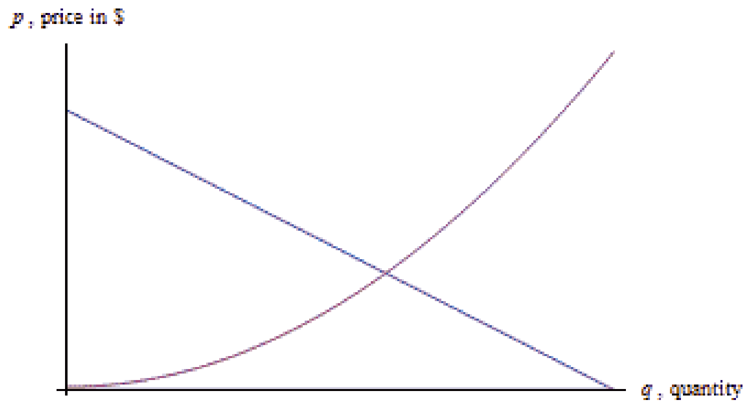
5.7 - APPLICATIONS TO BUSINESS – DEFINITE INTEGRATION

Consumer and Producer Surplus

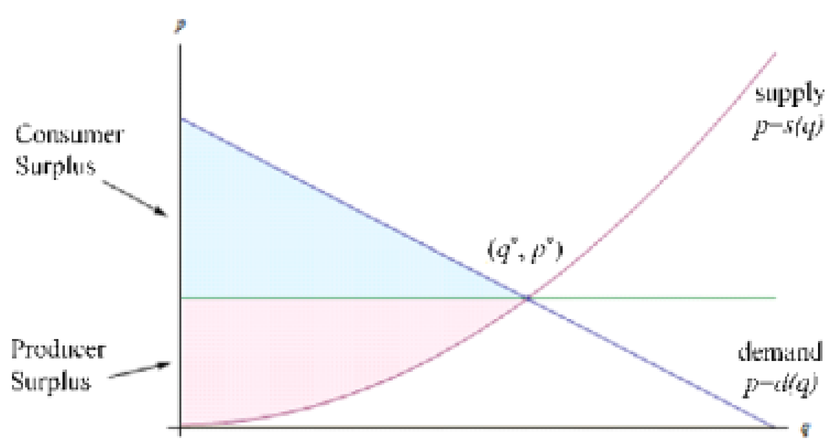
Here are a demand and a supply curve for a product. Which is which?



The demand curve is decreasing – lower prices are associated with higher quantities demanded, higher prices are associated with lower quantities demanded. Demand curves are often shown as if they were linear, but there's no reason they must be.

The supply curve is increasing – lower prices are associated with lower supply, and higher prices are associated with higher quantities supplied.

The point where the **"Demand"** and **"Supply"** curve cross is called the equilibrium point (q^*, p^*).



Suppose that the price is set at the equilibrium price, so that the quantity demanded equals the quantity supplied.

Now think about the folks who are represented on the left of the equilibrium point.

The consumers on the left would have been willing to pay a higher price than they ended up having to pay, so the equilibrium price saved them money.

On the other hand, the producers represented on the left would have been willing to supply these goods for a lower price – they made more money than they expected to.

Both groups ended up with extra cash in their pockets!

Let's think of this scenario:

Say that you want to buy a new car and are willing to pay $20,000, but the car you want actually costs $17,000.

Then, in a way you have saved $3,000. On the other hand, maybe the dealer would have been willing to sell the car for $15,000.

This means that the dealer has also gained $2,000 which he might have otherwise lost.

Both the consumer and the dealer are winning from an economy in which there is a competition on the price.

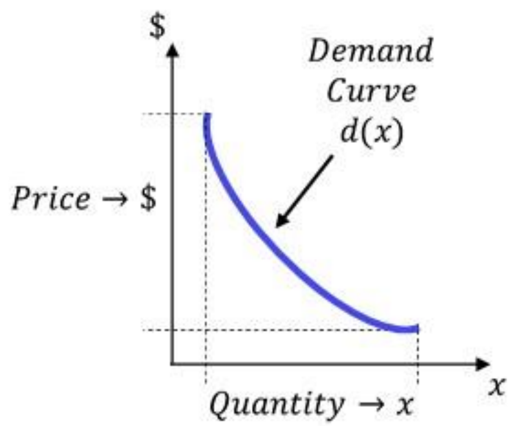
Consumer Surplus

For the cost of any product, if you were to add up the price that each consumer was willing to pay for that product minus the price paid, the total savings would be called the **"Consumers' Surplus"** for that product.

The **"Consumers' Surplus"** measures the benefit that consumers receive from an economy, in which competition keeps prices low.

Mathematically, **price** and **quantity** are inversely related: if the price of an item rises, the quantity sold generally falls, and vice versa.

This inverse relationship between price and quantity is expressed mathematically as a demand function or the demand curve ($d(x)$); which gives the price at which exactly (x) units will be demanded. See curve demand curve ($d(x)$) below.



The demand curve predicts the price that consumers are willing to pay for an item.

The market price is value consumers do pay for the item.

Thus, the amount in which the demand curve is above the market price, is the measure of the benefit or "surplus" to consumers.

If we think of the total accumulation of money saved by consumers who would have been willing to pay more than the market price, we obtain graphically the area between the demand curve and the horizontal line which is the market price.

Thus, the amount in which the demand curve is above the market price, is the measure of the benefit or "surplus" to consumers.

This total benefit is called the **Consumers' Surplus**. Mathematically, we can use integration, by adding up these benefits, so that the area between the demand curve and the market price line, gives the total benefit to consumers.

The mathematical definition of Consumer's Surplus is given below.

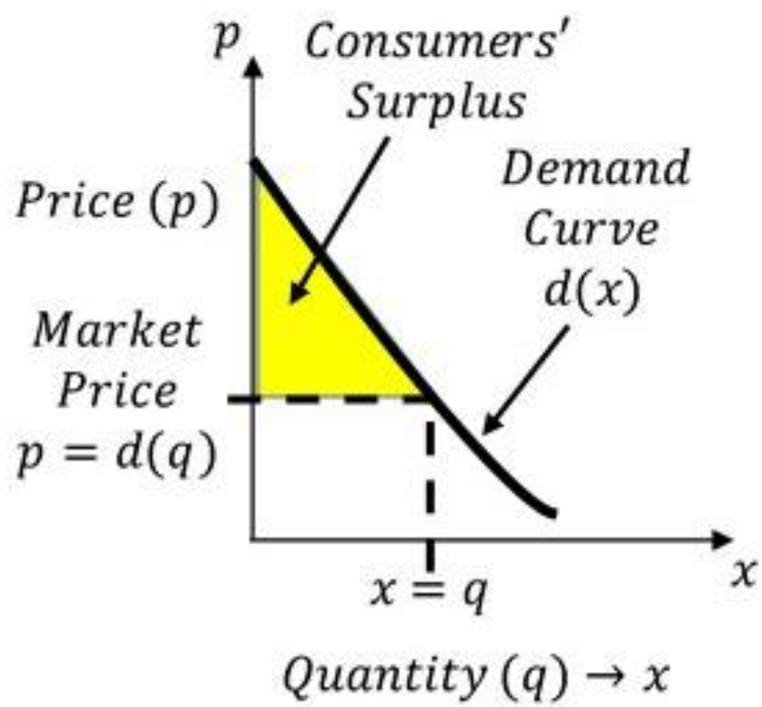
Consumer Surplus

For a **Demand** function ($d(x)$) and **Demand Level** (q), the **Market Price** (p) is the demand evaluated at ($x = q$), so that ($p = d(q)$).

The **consumer surplus** is the area between the demand curve and the market price.

$$Consumers'Surplus = \int_0^q [d(x) - p]dx$$

$$Consumers'Surplus = \int_0^q [d(x) - d(q)]dx$$



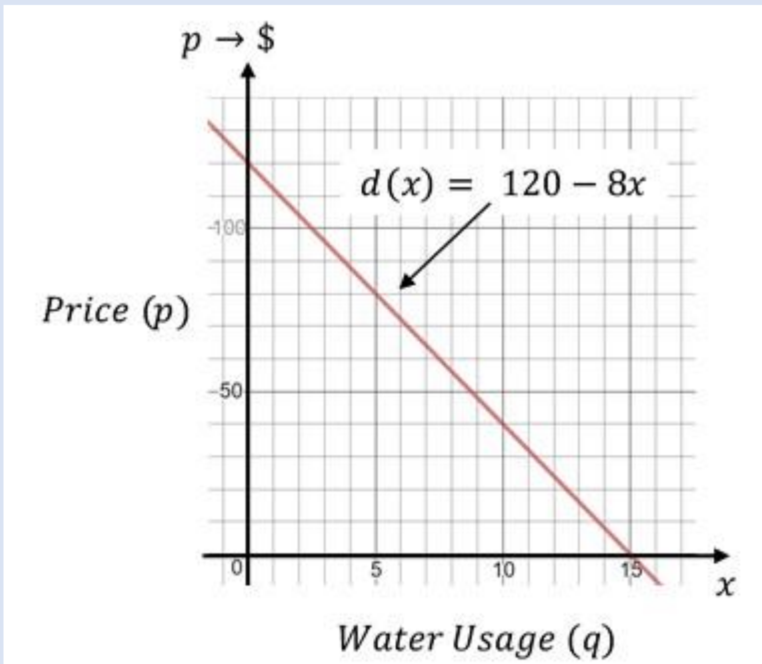
Example Problem #1

Finding Consumers' Surplus

A small city is supplying water to its population of 40,000 people. If the demand function for water is [$d(x) = 120 - 8x$] dollars,

where (x) is the water usage in millions of gallons, ($0 \leq x \leq 15$).

Find the consumers' surplus at the demand level [$x = 6$].



Solution:

First, find the Market Price (p), which is the Demand ($d(q)$) function evaluated the demand level:

$$[x = q = 6]$$

$$Market\ Price = p = d(q)$$

$$d(6) = 120 - 8(6) = \$72$$

Example Problem #1 – Cont'd

Finding Consumer's Surplus

The consumers' surplus is the area between the demand curve, and the market price line.

$$Consumers'Surplus = \int_0^q [d(x) - d(q)]dx$$

$$Consumers'Surplus = \int_0^6 (120 - 8x - 72)\,dx$$

$$Consumers'Surplus = \int_0^6 (48 - 8x)\,dx$$

$$Consumers'Surplus = [48x - 4x^2]_0^6$$

$$Consumers'Surplus = [48(6) - 4(6)^2] - [0 - 0]$$

$$Consumers'Surplus = [480 - 144] = 144$$

$$\boldsymbol{Consumers'Surplus} = \$144$$

Therefore, the consumers' surplus for water usage is $144.

If water usage were to increase to [$x = 10$], the market price would then drop to [$d(10) = 120 - 80 = \$40$].

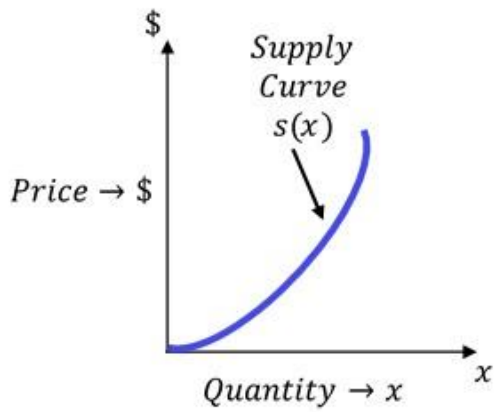
That means that if the price decreases from $72 to $40, the consumers would benefit by an additional $336 – $144 = $192.

Producer Surplus

The producers also benefit from a market economy in which prices are competitive. The **Producers' Surplus** measures the total benefit that producers get from selling an item at the market price.

Just as the consumers might have been willing to pay a higher price for an item, the producers might have been willing to charge less for an item. The extra money the consumers spend for an item is surplus money in the pockets of the suppliers.

As the price increases, the producers are willing to supply more items at that higher price, so the supply function is increasing with respect to price.



If we think of the total accumulation of money saved by the dealer, or the producer who would have been willing to lower the price even more than the market price, we obtain graphically the area between the supply curve and the horizontal line which is the market price.

Thus, the amount in which the supply curve is below the market price, is the measure of the benefit or "surplus" to producers.

This total benefit is called the **Producers' Surplus**. Mathematically, we can use integration, by adding up these benefits, so that the area between the supply curve and the market price line, gives the total benefit to producers.

The mathematical definition of Producer's Surplus is given below.

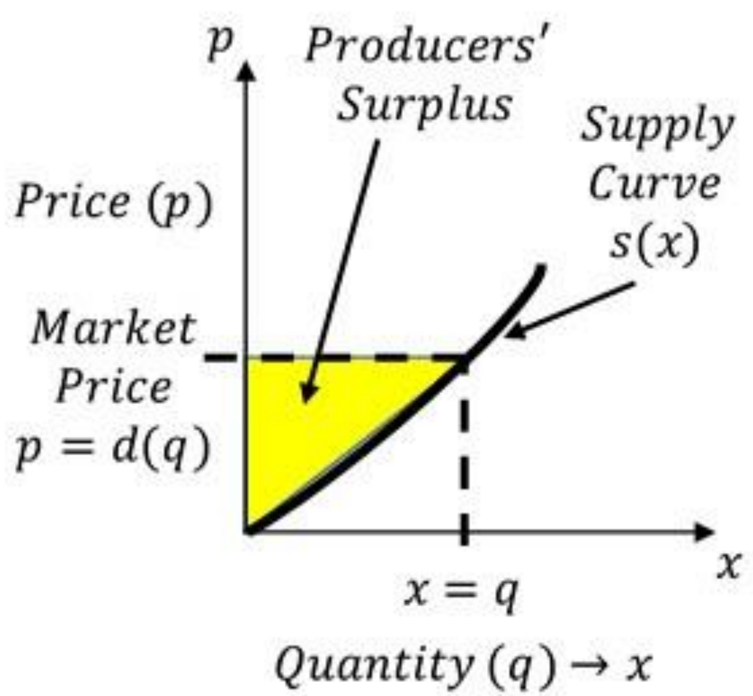
Producer Surplus

For a **Supply** function ($s(x)$) and **Demand Level** (q), the **Market Price** (p) is the demand evaluated at ($x = q$), so that ($p = s(q)$).

The **producer surplus** is the area between the market price and the supply curve.

$$Producers'Surplus = \int_0^q [p - s(x)]dx$$

$$Producers'Surplus = \int_0^q [s(q) - s(x)]dx$$

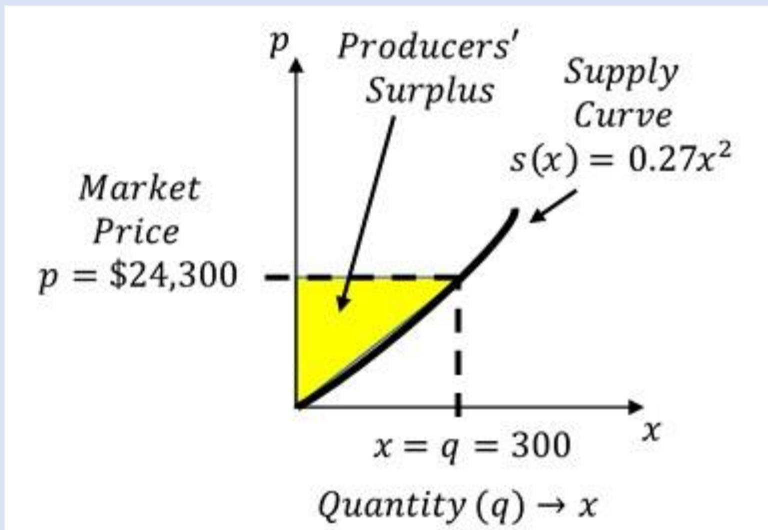


Example Problem #2

Finding Producer's Surplus

A small city is supplying water to its population of 40,000 people according to the **Supply** function [$s(x) = 0.27x^2$] dollars, where (x) is the water usage in millions of gallons, ($0 \le x \le 15$).

Find the producers' surplus at the demand level ($x = 300$).



Solution:

The market price is the **supply** function ($s(x)$) evaluated at the demand level of

$$[x = q = 300]$$

$$Market\ Price = p = s(q)$$

$$Market\ Price = s(300) = 0.27(300)^2 = 24{,}300$$

$$Producers'Surplus = \int_0^q [s(q) - s(x)]dx$$

Example Problem #2 - Cont'd

Finding Producer's Surplus

$$Producers' Surplus = \int_0^{300} [24300 - 0.27x^2]dx$$

$$Producers' Surplus = \left[24300x - \frac{0.27}{3}x^3\right]_0^{300}$$

$$Producers' Surplus = [24300(300) - 0.09(300)^3]$$

$$Producers' Surplus = [7{,}290{,}000 - 2{,}430{,}000]$$

$$\boldsymbol{Producers' Surplus} = \$4{,}860{,}000$$

The producers' surplus is $4,860,000.

Combined Consumer's Surplus and Producer Surplus

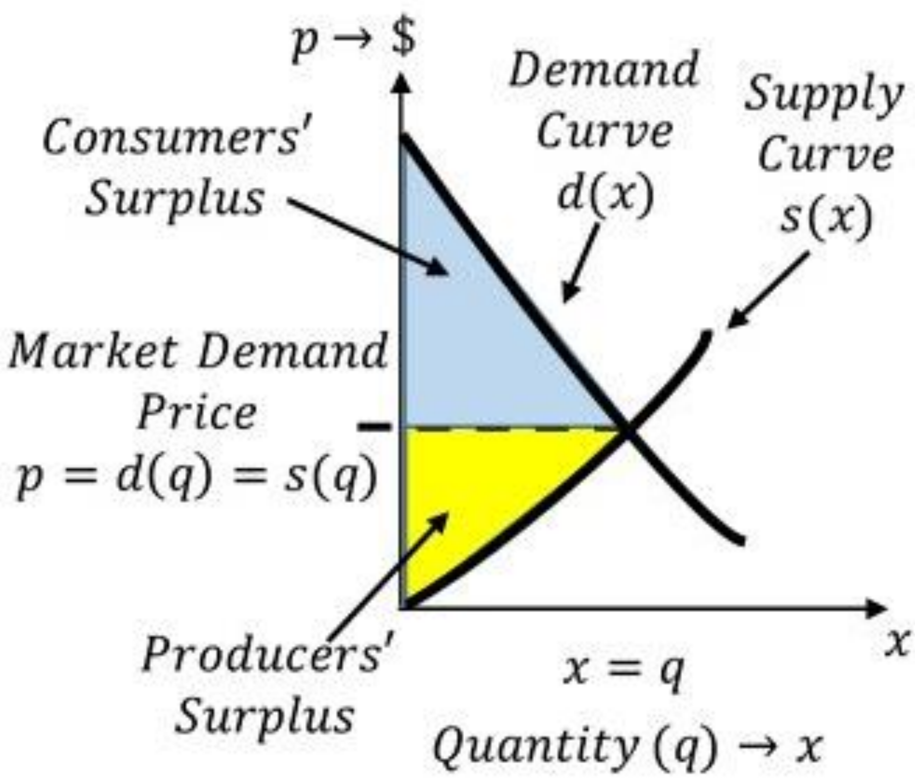
The Market Price (p) point of demand, at which the supply ($s(x)$) and demand ($d(x)$) curves intersect is called the **market demand** (p).

The consumers' surplus and the producers' surplus can be shown together on the same graph.

These two areas together give a numerical measure of the total benefit that consumers and producers get from competition.

This shows that both consumers and producers benefit from an open market.

The point where the **"Demand"** and **"Supply"** curve cross is called the equilibrium point $(p = d(q) = s(q))$.



Example Problem #3

Finding Consumers' Surplus and Producers' Surplus

Given the demand function $d(x)) = (4 - x)^2$ and the supply function $s(x) = x^2 + 2x + 6$, both in thousands of dollars.

a. Find the equilibrium point.

b. Find the consumers' surplus at the market demand found in part (a).

c. Find the producers' surplus at the market demand found in part (a).

Solution:

The Equilibrium point is reached by setting the two functions equal and solve.

$$d(x) = s(x)$$

$$(4 - x)^2 = x^2 + 2x + 6$$

$$x^2 - 8x + 16 = x^2 + 2x + 6$$

$$10 = 10x$$

$$x = 1$$

Let the quantity x - value equal ($q = x = 1$) where the Market Price (p) becomes the following value:

$$p = d(q) = s(q) = (4 - x)^2$$

$$p = d(1) = s(1) = (1 - x)^2 = 9$$

$$p = d(q) = 9$$

The equilibrium point is (1, 9).

Example Problem #3 – Cont'd

Finding Consumer's Surplus

$$Consumers'Surplus = \int_0^q [d(x) - d(q)]dx$$

$$Consumers'Surplus = \int_0^1 ((4 - x)^2 - 9)\,dx$$

$$Consumers'Surplus = \int_0^1 (x^2 - 8x + 7)\,dx$$

$$Consumers'Surplus = \left[\frac{x^3}{3} - 4x^2 + 7x\right]_0^1$$

$$Consumers'Surplus = \left[\frac{1}{3}(1) - 4(1)^2 + 7(1)\right] - [0 - 0 + 0]$$

$$\boldsymbol{Consumers'Surplus} = 3.33\ thousand\ of\ dollars.$$

Finding Producer's Surplus

$$Producers'Surplus = \int_0^q [s(q) - s(x)]dx$$

$$Producers'Surplus = \int_0^1 (9 - (x^2 + 2x + 6))\,dx$$

$$Producers'Surplus = \int_0^1 (3 - x^2 - 2x)\,dx$$

$$Producers'Surplus = \left[3x - \frac{1}{3}x^3 - x^2\right]_0^1$$

$$Producers'Surplus = \left[3(1) - \tfrac{1}{3}(1) - (1)^2\right] - [0 - 0 - 0]$$

$$\boldsymbol{Producers'Surplus} = 1.6667\ thousands\ of\ dollars.$$

Continuous Income Stream or Continuous Money flow

In College Algebra, you learned about compound interest, typically in that simple situation where you made a single deposit into an interest-bearing account, and let it sit undisturbed, earning interest, for some period, of time. Recall.

Compound Interest Formulas

Let (P) be the **Principal** (initial investment), (r) be the **Annual Interest Rate** expressed as a decimal, and ($A(t)$) be the amount in the account at the end of (t) years.

Compounding (n) times per year

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Compounding Continuously

$$A(t) = Pe^{rt}$$

If you're using this formula to find what an account will be worth in the future:
($t > 0$) and ($A(t)$) is called the "**Future Value**".

If you're using the formula to find what you need to deposit today, to have a certain value (P) sometime in the future:
($t < 0$) and ($A(t)$) is called the "**Present Value**".

You may also have learned somewhat more complicated annuity formulas to deal with slightly more complicated situations – where you make equal deposits equally spaced in time.

But real-life is not usually so neat. Calculus allows us to handle situations where deposits are flowing continuously into an account that earns interest.

As, long as, we can model the flow of income with a function, we can use a definite integral to calculate the amount of money we will have in the account at the end of a period of time.

This function that describes the amount of money invested in the account each year is called a continuous stream of income (as it keeps coming in every year). It is also called in some publications a continuous money flow.

We can also compute the amount of money that would need to be invested in the present to create this yearly continuous stream of income in the future. We call this amount present value.

The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then add them all up (a definite integral). This is a useful way to compare two investments – find the present value of each to see which is worth more today.

Present Value of a Continuous Stream of Income (Money Flow)

Suppose money can earn interest at an **Annual Interest Rate** of (r), compounded continuously.

Let ($C(t)$) be a **Continuous Income Stream, also called Continuous Money Flow** function (in dollars per year) that applies between year (0) and year (T).

The **Present Value** ($P_V(T)$) of a **Continuous Stream of Income** ($C(t)$) dollars per year, where (t) is the number of years from now, for (T) years, at continuous interest (r)

$$Present\ Value = P_V(T) = \int_0^{T\ (years)} C(t)e^{-rt}\,dt$$

The **Future Value** ($P_F(T)$) of a **Continuous Stream of Income** ($C(t)$) dollars per year will be

$$Future\ Value = P_F(T) = \int_0^{T\ (years)} C(t)e^{rt}\,dt$$

If, $C(t) = Constant$, i.e it is a fixed **same** amount each year, then it can be proven that

$$P_F(T) = P_V(T)e^{rT}$$

Example Problem #4

You have an opportunity to buy a business that will earn $75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously.

Is this business worth its purchase price of $630,000?

Solution:

We must assume that the interest rates are going to remain constant for that entire eight years.

We also must assume that the $75,000 per year is coming in continuously, like a faucet dripping dollars into the business.

Neither of these assumptions might be accurate.

But moving on: The present value of the $630,000 is, well, $630,000.

This is one investment, where we put our $630,000 in the bank and let it sit there.

To find the present value of the business, we think of it as an income stream.

The function ($C(t)$) **in this case is a constant** $75,000 dollars per year.

The interest rate is (r = 2.8%) and the term we're interested in is (T = 8) years, so

$$C(t) = \$75000 \quad ; \qquad r = 0.028 \;; \qquad T = 8$$

Present Value Solution:

$$Present\ Value = P_V(T) = \int_0^{T\ (years)} C(t)e^{-rt}\,dt$$

$$Present\ Value = P_V(T) = \int_0^{8} 75000e^{-0.028t}\,dt$$

$$P_V(T) = 75000\left[\frac{e^{-0.028t}}{-0.028}\right]_0^8 = -\frac{75000}{0.028}\left[e^{-0.028(8)} - 1\right]$$

$$P_V(T) \cong \$537{,}548$$

Example Problem #4 – Cont'd

The **Present Value** of the business is about $537,500, which is less than the $630,000 asking price, so this is not a good deal, not worth buying.

Future Value Solution:

$$r = 0.028 \; ; \qquad T = 8$$

$$Future\ Value = P_F(T) = P_V(T)e^{rt}$$

$$Future\ Value = P_V(T) = 537{,}548\left(e^{0.028(8)}\right)$$

$$P_F(T) \cong \$672,511$$

However, when considering the **Future Value** of the business, the business is worth about $672,511, which is greater than the $630,000 asking price.

So, if you were considering eight (8) years, into the future, then buying the business for the asking price, would be worth it!

5.7 - EXERCISES

For each demand function ($d(x)$) and demand level (x), find the Consumers' Surplus			
1.	$d(x) = 2600 - 7x$ $x = 150$	2.	$d(x) = 100 - 0.2x$ $x = 75$
3.	$d(x) = 75e^{-0.03x}$ $x = 200$	4.	$d(x) = -0.5x^2 + 225$ $x = 16$
For each Supply function ($s(x)$) and demand level (x), find the Producers' surplus			
5.	$s(x) = 0.05x$ $x = 250$	6.	$s(x) = 0.18x^2$ $x = 300$
For each demand function ($d(x)$) and supply function ($s(x)$): c. Find the market demand (the positive value of (x) at which the demand function intersects the supply function). d. Find the consumers' surplus at the market demand found in part (a). e. Find the producers' surplus at the market demand found in part (a).			
7.	$d(x) = 600 - 0.8x$ $s(x) = 0.4x$	8.	$d(x) = 400 - 0.4x^2$ $s(x) = 0.6x^2$

Determine the Continuous Income Stream Problems.	
9.	Find the present and future values of a continuous income stream of \$5000 per year for 12 years if money can earn 1.3% annual interest compounded continuously.
10.	Find the present value of a continuous income stream of \$40,000 per year for 35 years if money can earn (a) 0.8% annual interest, compounded continuously, (b) 2.5% annual interest, compounded continuously, (c) 4.5% annual interest, compounded continuously.
11.	A business is expected to generate income at a continuous rate of \$25,000 per year for the next eight (8) years. Money can earn 3.4% annual interest, compounded continuously. The business is for sale for \$153,000. Is this a good deal?

Solutions

65. $\$78{,}750$

66. $\$562.50$

67. $\$2{,}456.62$

68. $\$1.365.33$

69. $\$1{,}562.50$

70. $\$3{,}240{,}000$

71. $a.\quad x = 500,\qquad b.\quad \$100{,}000\qquad c.\quad \$50{,}000$

72. $a.\quad x = 20,\qquad b.\quad \$2133.33,\qquad c.\quad \$3200$

73. $PV = \$55{,}554.16\quad ;\quad FV = \$64{,}933.16$

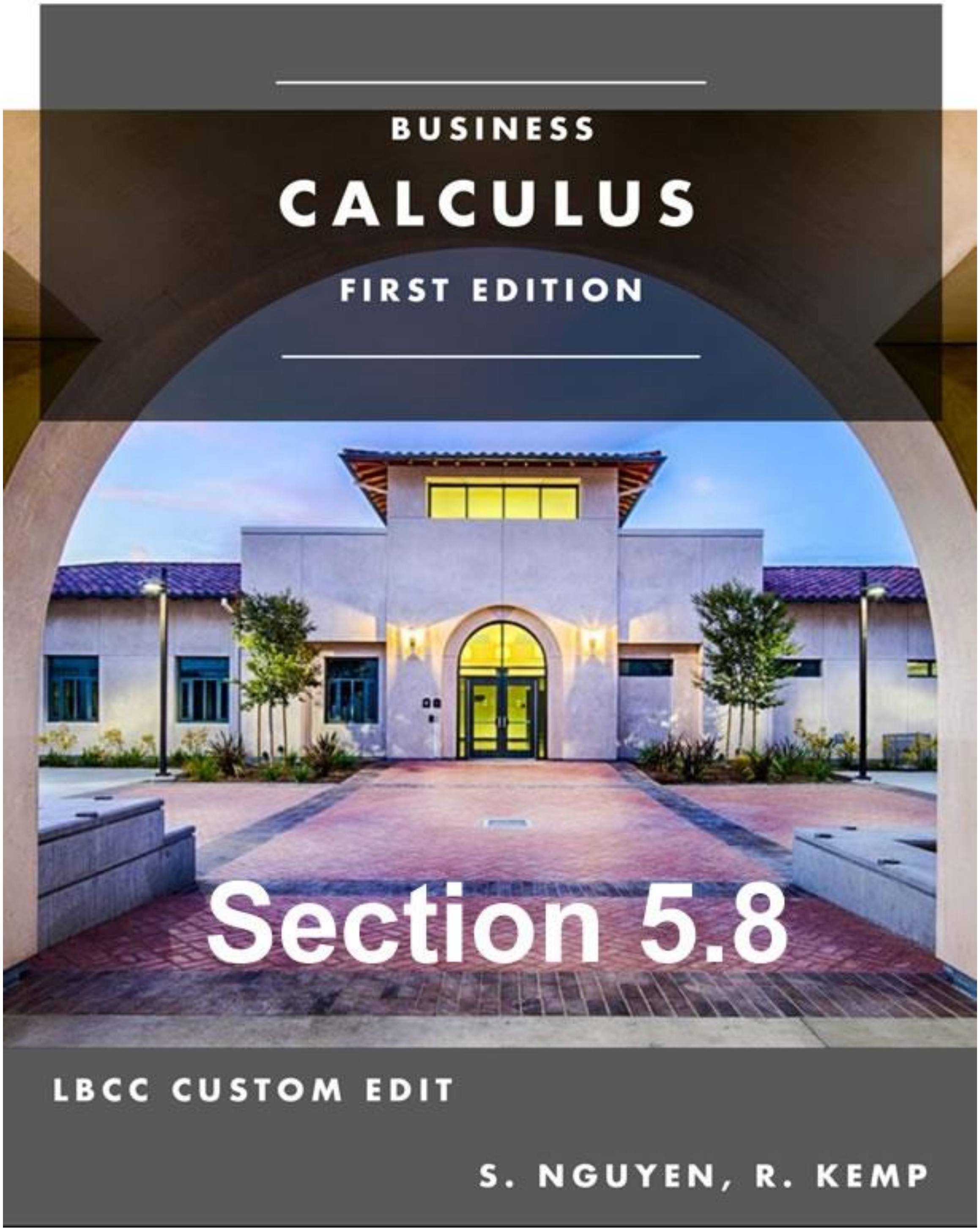
74. $a.\qquad PV = \$1{,}221{,}081\qquad FV = \$1{,}615{,}649$

    $b.\qquad PV = \$933{,}020\qquad FV = \$2{,}238{,}200$

    $c.\qquad PV = \$704{,}822\qquad FV = \$3{,}405{,}103$

75. $PV = \$175{,}107\quad ;\quad FV = \$229{,}843$

    $PV = \$175{,}107 > \$153{,}000\ \ therefore\ buy\ the\ business$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5.8 - SUBSTITUTION INTEGRATION

Introduction

We do not have many integration rules. For quite a few of the problems we see, the rules will not directly apply; we'll have to do some algebraic manipulation first. In practice, it is much harder to write down the antiderivative of a function than it is to find a derivative.

In fact, it's very easy to write a function that doesn't have any antiderivative you can find with algebra, although proving that it doesn't have an antiderivative is much more difficult.

The Substitution Method (also called U-Substitution) is one way of algebraically manipulating an integrand so that the rules apply. This is a way to unwind or undo the Chain Rule for derivatives.

When you find the derivative of a function using the Chain Rule, you end up with a product of something like the original function times a derivative. We can reverse this to write an integral:

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x)$$

So,

$$f\big(g(x)\big) = \int \big[f'\big(g(x)\big)\cdot g'(x)\big]\,dx$$

With substitution, we will substitute,

$$u = g(x) \qquad \text{and} \qquad f\big(g(x)\big) = f(u)$$

This means,

$$\frac{du}{dx} = g'(x) \qquad \text{and} \qquad du = g'(x)dx$$

Making the above substitutions, makes it easier to integrate, as in the following.

$$\int f'\big(g(x)\big)\cdot g'(x)\cdot dx = \int f'(u)\cdot du$$

The Substitution Method for Antiderivatives is to turn,

$$\int f\big(g(x)\big)\cdot g(x)dx$$

Into

$$\int f(u)du,$$

where $(f(u))$ is much less messy than $(f(g(x)))$.

The Substitution Method for Antiderivatives:

4. Let $(u = g(x))$, be some part of the integrand.

5. Compute: $du = \left(\frac{du}{dx}\right) dx$

6. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx).

   - When you're done, you should have a new integral that is entirely in (u).

   - If you have any (x)'s remaining, then that's an indication that the substitution didn't work or isn't complete; you may need to go back to step 1, and try a different choice for (u).

7. Integrate the new u-integral, if possible.

   - If you are still unable to integrate it, go back to step 1 and try a different choice for (u).

8. Finally, substitute back (x)'s for (u)'s everywhere in your answer.

When you use substitution to help evaluate an **"Indefinite integral"**, you must add the constant "K" after the integration step.

Example Problem #1

Find the indefinite integral.

$$\int \frac{x}{\sqrt{4-x^2}}dx$$

Solution:

This integrand is more complicated than anything in our list of basic integral formulas, so we will have to try something else. The only tool we currently have to use is the substitution method.

1. Let (u) be some part of the integrand:

$$u = 4 - x^2.$$

2. Next, compute: $du = \left(\frac{du}{dx}\right) dx$

$$du = -2xdx$$

There is (xdx) in the integrand:

$$\left(-\frac{du}{2}\right) = xdx$$

3. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx):

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx = \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{du}{2}\right) = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx = -\frac{1}{2}\int u^{-\frac{1}{2}}\, du$$

4. Integrate the new (u)-integral, if possible:

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx = -\frac{1}{2}\left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + K = -\frac{1}{2}\left[\frac{\sqrt{u}}{\frac{1}{2}}\right] + K$$

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx = -\sqrt{u} + K$$

Example Problem #1 – Cont'd

5. Finally, substitute back (x)'s for (u)'s everywhere in the answer:

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx \;=\; -\sqrt{u} \;+\; K$$

Where: $u = 4 - x^2$

The final answer is the following:

$$\int \frac{x}{\sqrt{4-x^2}} \cdot dx \;=\; -\sqrt{4 - x^2} \;+\; K$$

6. The above final answer can be checked; by differentiating:

$$\frac{d}{dx}\left(-\sqrt{4 - x^2} \;+\; K\right) \;=\; \frac{d}{dx}\left(-(4 - x^2)^{\frac{1}{2}} \;+\; K\right)$$

$$\frac{d}{dx}\left(-\sqrt{4 - x^2} \;+\; K\right) \;=\; -\frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\frac{d}{dx}\left(-\sqrt{4 - x^2} \;+\; K\right) \;=\; x \cdot (4 - x^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left(-\sqrt{4 - x^2} \;+\; K\right) \;=\; \frac{x}{\sqrt{4-x^2}}$$

Example Problem #2

Find the indefinite integral.

$$\int \frac{e^x}{(e^x + 15)^3} \cdot dx$$

Solution:

1. Let (u) be some part of the integrand:

$$u = (e^x + 15).$$

2. Next, compute: $$du = \left(\frac{du}{dx}\right) dx$$

There is $(e^x dx)$ in the integrand:

$$du = e^x dx$$

3. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx):

$$\int \frac{e^x}{(e^x + 15)^3} \cdot dx = \int \frac{1}{(u)^3} \cdot du = \int u^{-3} \, du$$

4. Integrate the new (u)-integral, if possible:

$$\int \frac{e^x \cdot dx}{(e^x + 15)^3} = \left[\frac{u^{-3+1}}{-3 + 1}\right] + K = -\frac{1}{2}[u^{-2}] + K = -\frac{1}{2u^2} + K$$

5. Finally, substitute back (x)'s for (u)'s everywhere in the answer:

$$\int \frac{e^x}{(e^x + 15)^3} \cdot dx = -\frac{1}{2u^2} + K$$

Where: $$u = (e^x + 15)$$

The final answer is the following:

$$\int \frac{e^x}{(e^x + 15)^3} \cdot dx = -\frac{1}{2(e^x + 15)^2} + K$$

Example Problem #3

Find the indefinite integral.

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx$$

Solution:

1. Let (u) be some part of the integrand:

$$u = (x^3 + 5)$$

2. Next, compute: $$du = \left(\frac{du}{dx}\right) dx = 3x^2 dx$$

There is $(x^2 dx)$ in the integrand: $$\left(\frac{du}{3}\right) = x^2 dx$$

3. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx):

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx = 2\int (x^3 + 5)^3 (x^2 \cdot dx)$$

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx = 2\int (u)^3 \left(\frac{du}{3}\right) = \frac{2}{3}\int u^3 du$$

4. Integrate the new (u)-integral, if possible:

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx = \frac{2}{3}\left[\frac{u^{3+1}}{3 + 1}\right] + K = \frac{1}{6}(u^4) + K$$

5. Finally, substitute back (x)'s for (u)'s everywhere in the answer:

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx = \frac{1}{6}(u^4) + K$$

Where: $$u = (x^3 + 5)$$

The final answer is the following:

$$\int 2x^2 \cdot (x^3 + 5)^3 \cdot dx = \frac{1}{6}(x^3 + 5)^4 + K$$

Example Problem #4

A bacteria culture starts with 800 bacteria.

Knowing that the bacteria is multiplying at a rate of $\left(\frac{x^4}{x^5-1}\right)$ bacteria per hour, where x = number of hours since the culture started, find a formula for the number of bacteria x hours later.

Solution:

The number of bacteria x hours later is the integral of its rate of growth.

$$B(x) = \int \frac{x^4}{x^5 - 1} \cdot dx$$

1. Let (u) be some part of the integrand:

$$u = (x^5 - 1).$$

2. Next, compute: $du = \left(\frac{du}{dx}\right) dx = 5x^4 dx$

There is $(x^3 dx)$ in the integrand:

$$\frac{du}{5} = x^4 dx$$

3. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx):

$$B(x) = \int \frac{x^4}{x^5 - 1} \cdot dx = \frac{1}{5}\int \frac{1}{u} \cdot du = \frac{1}{5}\int u^{-1}\ du$$

4. Integrate the new (u)-integral, if possible:

$$B(x) = \int \frac{x^4}{x^5 - 1} \cdot dx = \frac{1}{5} ln|u| + K$$

5. Finally, substitute back (x)'s for (u)'s everywhere in the answer:

$$B(x) = \int \frac{x^4}{x^5 - 1} \cdot dx = \frac{1}{5} ln|x^5 - 1| + K$$

Example Problem #4 -Cont'd

In the above result, we can drop the absolute value bars in the natural logarithm function because the value ($x^5 - 1$) is positive for every value of (x).

$$B(x) = \frac{1}{5} ln|x^5 - 1| + K$$

6. To evaluate the constant (K), we set the function, evaluated at ($x = 0$), equal to its initial amount of 800.

$$B(0) = \frac{1}{5} ln|-1| + K = 800$$

Thus, the constant (K) is:

$$K = 800$$

7. Replace the constant ($K = 800$) value into the function $B(x)$ and evaluate.

The amount of bacteria function ($B(x)$) is:

$$B(x) = \frac{1}{5} ln|x^5 - 1| + 800$$

Substitution and Definite Integrals

When you use substitution method to help evaluate a "Definite Integral", you have a choice for how to handle the limits of integration. You have the option of changing the limits to match the variable of integration.

Thus, you can change the limits to match the substitution variable. Or you can keep the original limits, if you replace the original variable for the substituted one, after the process of integration.

You can use either method, whichever seems better to you. The important thing to remember is – the original limits of integration were values of the original variable (say, ***x***), not values of the new variable (say, ***u***).

(a) You can solve the antiderivative as a side problem, translating back to x's, and then use the antiderivative with the original limits of integration.

(b) You can substitute for the limits of integration at the same time as you're substituting for everything inside the integral, and then skip the "translate back into x" step.

If the original integral had endpoints: [x = a] and [x = b]

$$\int_{x=a}^{x=b} f(g(x)) \cdot dx$$

And we make the substitution: $[u = g(x)]$, and $[du = g'(x)]$;

Then the new integral will have endpoints

$$[u(a) = g(a)], \quad \text{and} \quad [u(b) = g(b)]$$

$$\int_{x=a}^{x=b} f(g(x)) \cdot dx \quad = \quad \int_{u(a)}^{u(b)} f(u) \cdot du$$

Method (a) seems more straightforward for most students. But it can involve some messy algebra. Method (b) is often neater and usually involves fewer steps.

Example Problem #5

Evaluate:

$$\int_0^1 (3x - 1)^4 \cdot dx$$

Solution:

1. Let (u) be some part of the integrand:

$$u = (3x - 1)$$

2. Next, compute: $du = \left(\frac{du}{dx}\right) dx = 3\, dx$

There is (dx) in the integrand: $\left(\frac{du}{3}\right) = dx$

3. Change the Limits of Integration to match the substitution variable:

$$u\,(a) = u\,(0) = [3(0) - 1] = -1$$

$$u\,(b) = u\,(1) = [3(1) - 1] = 2$$

4. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx), and changing the limits of integration:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{3}\int_0^1 (u)^4 \cdot du = \frac{1}{3}\int_{-1}^2 (u)^4 \cdot du$$

5. Integrate the new (u)-integral, if possible:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{3}\left[\frac{u^5}{5}\right]_0^1 = \frac{1}{3}\left[\frac{u^5}{5}\right]_{-1}^2$$

6. On the left side, substitute back (x)'s for (u)'s in the above result:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}[(3x - 1)^5]_0^1 = \frac{1}{15}[u^5]_{-1}^2$$

Where: $u = (3x - 1)$

Example Problem #5 – Cont'd

7. Next, plug in the **[x]** limits of integration into the results above:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}[(3x - 1)^5]_0^1$$

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}\left[[(3(1) - 1)^5] - [(3(0) - 1)^5]\right]$$

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}\left[[(2)^5] - [(-1)^5]\right]$$

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{5}[32 + 1]$$

Final answer:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{33}{15}\ square\ units$$

8. Next, plug in the **[u]** limits of integration into the results above:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}[u^5]_{-1}^2$$

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{15}\left[[(2)^5] - [(-1)^5]\right]$$

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{1}{5}[32 + 1]$$

Final answer:

$$\int_0^1 (3x - 1)^4 \cdot dx = \frac{33}{15}\ square\ units$$

Example Problem #6

Evaluate:

$$\int_{2}^{10} \frac{(\ln(x))^6}{x} \cdot dx$$

Solution:

1. Let (u) be some part of the integrand:

$$u = \ln(x)$$

2. Next, compute: $$du = \left(\frac{du}{dx}\right) dx = \frac{1}{x}\, dx$$

There is (dx) in the integrand: $$du = \frac{1}{x} dx$$

3. Change the Limits of Integration to match the substitution variable:

$$u\,(a) = u\,(2) = \ln(2)$$

$$u\,(b) = u\,(10) = \ln(10)$$

4. Translate all your (x)'s into (u)'s everywhere in the integral, including the (dx), and changing the limits of integration:

$$\int_{2}^{10} \frac{(\ln(x))^6}{x} \cdot dx = \int_{2}^{10} (u)^6 \cdot du = \int_{\ln(2)}^{\ln(10)} (u)^6 \cdot du$$

5. Integrate the new (u)-integral, if possible:

$$\int_{2}^{10} \frac{(\ln(x))^6}{x} \cdot dx = \left[\frac{u^7}{7}\right]_{2}^{10} = \left[\frac{u^7}{7}\right]_{\ln(2)}^{\ln(10)}$$

6. On the left side, substitute back (x)'s for (u)'s in the above result:

$$\int_{2}^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[(\ln(x))^7]_{2}^{10} = \frac{1}{7}[u^7]_{\ln(2)}^{\ln(10)}$$

Where: $$u = \ln(x)$$

Example Problem #6 – Cont'd

7. Next, plug in the **[x]** limits of integration into the results above:

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[(\ln(x))^7]_2^{10}$$

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[(\ln(10))^7 - (\ln(2))^7]$$

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[7 \cdot (\ln(10)) - 7 \cdot (\ln(2))]$$

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \ln\left(\frac{10}{2}\right)$$

Final answer:

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx \cong \ln(5) = 1.609$$

8. Next, plug in the **[u]** limits of integration into the results above:

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[u^7]_{\ln(2)}^{\ln(10)}$$

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx = \frac{1}{7}[(\ln(10))^7 - (\ln(2))^7]$$

Final answer:

$$\int_2^{10} \frac{(\ln(x))^6}{x} \cdot dx \cong \ln(5) = 1.609$$

5.8 - EXERCISES

Find each indefinite Integral by the Substitution Method			
1.	$\int (1 + x^4)^3\, 4x^3 dx$	2.	$\int \left(\frac{1}{2}x^2 + 1\right)^5 x\,dx$
3.	$\int \frac{x^3 dx}{1 + x^4}$	4.	$\int x^5 e^{x^6}\,dx$
5.	$\int 5e^{5y}\,dy$	6.	$\int x^2(9 - x^3)^4\,dx$
7.	$\int \frac{dx}{2 - 4x}$	8.	$\int \frac{2x}{1 + x^2} dx$
9.	$\int (x + 1)(3x^2 + 6x)^6\,dx$	10.	$\int \frac{3e^{3x}}{3 + e^{3x}} dx$
11.	$\int \frac{(x^5 + x^4)}{(5x^6 + 6x^5)^7}\,dx$	12.	$\int (6 - 12y)^2\,dy$

13.	A company's Marginal Revenue function is ($MR(x)$) thousands of dollars, where x is the number of items sold; and the company make \$0 from selling 0 items. Find the Revenue function. $$MR(x) = \frac{5}{\sqrt{5x+1}}$$		
	Find each Definite Integral by the Substitution Method		
14.	$$\int_0^2 3e^{x^3}x^2\,dx$$	15.	$$\int_0^1 \frac{4x^3}{4 + x^4}\,dx$$
16.	$$\int_0^3 \left(\frac{1}{\sqrt{x^2 + 16}}\right) x\,dx$$		
17.	The population of a city is expected to be ($P(t)$) million people after (t) years. Find the Average population between year (t = 0) and year (t = 2). $$P(t) = \frac{3t^2}{t^3 + 2}$$		
18.	A Watch Maker's company marginal Cost (in thousands) during month (x) is given by ($MC(x)$). Find the total accumulated costs ($C(x)$) for the company, during the month (x = 4) to month (x = 20). $$MC(x) = \frac{5}{\sqrt{5 + x}}$$		

Solutions

76. $\frac{1}{4}(1 + x^4)^4 + K$

77. $\frac{1}{6}\left(\frac{1}{2}x^2 + 1\right)^6 + K$

78. $\frac{1}{4} ln|1 + x^4| + K$

79. $\frac{1}{6}e^{x^6} + K$

80. $e^{5y} + K$

81. $-\frac{1}{15}(9 - x^3)^5 + K$

82. $-\frac{1}{4} ln|2 - 4x| + K$

83. $ln|1 + x^2| + K$

84. $\frac{1}{42}(3x^2 + 6x)^7 + K$

85. $ln|3 + e^{3x}| + K$

86. $-\frac{1}{180}(5x^6 + 6x^5)^{-6} + K = \frac{-1}{180(5x^6 + 6x^5)^6} + K$

87. $-\frac{1}{36}(6 - 12y)^3 + K$

88. $Revenue\ function \rightarrow \quad R(x) = 2\sqrt{5x+1} - 2$

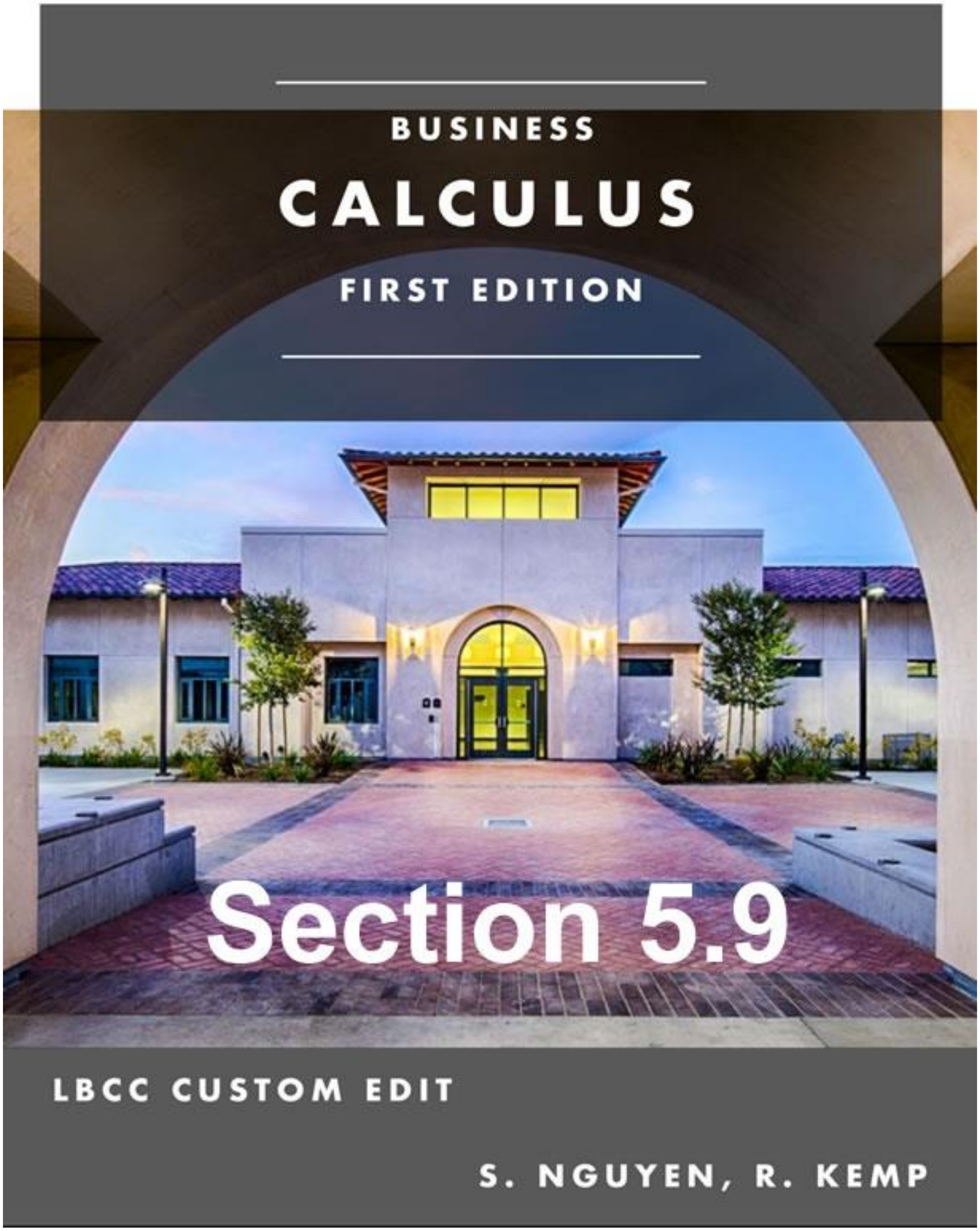
89. $e^8 - 1$

90. $ln\left(\frac{5}{4}\right)$

91. $5 - 4 = 1$

92. $\frac{1}{2}\ln 5 = 0.8047 \quad Million\ people$

93. $20\ Thousand\ dollars$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5.9 - INTEGRATION BY PARTS

Introduction

In this section, we will discuss the method of "Integration by Parts", which is the reverse process to the "Product Rule" for derivatives.

Integration by Parts

Integration by parts is an integration method which enables us to find antiderivatives of some new functions such as ($ln(x)$) as well as antiderivatives of products of functions such as ($x^2 ln(x)$) and (xe^x).

If the function we're trying to integrate can be written as a product of two functions, (u) and (dv), then the technique of "Integration by Parts" lets us trade out a complicated integral for hopefully a much simpler one.

Given two differentiable functions, ($u(x)$) and ($v(x)$), the Product Rule is for taking the derivative of their product (uv) is:

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Next expressing the above derivative ($\frac{d}{dx}$) as a differential (d).

$$d(uv) = udv + vdu$$

Next integrating both sides of the equation.

$$uv = \int d(uv) = \int udv + \int vdu$$

Next, solve the above equation for the first integral:

$$\int udv = uv - \int vdu$$

The “**Integration by Parts”**, mathematical formulation is given below.

Integration by Parts Formula:

For two differentiable functions, (u) and (v)

Indefinite Integral

$$\int u\,dv = uv - \int v\,du$$

Definite Integral

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

The challenge of the **“Integration by Parts**” method is to choose the (u) and the (dv), so that the resulting integral $(\int v\,du)$ is simpler than the original integral $(\int u\,dv)$.

As you see, in this method, to evaluate an integral, we have another integral to evaluate in the answer.

If this integral is as hard to evaluate as our original one, the method doesn't help.

How to Choose the (u) and (dv)

Sometimes selecting which term is the best choice for (u) and (dv) may involve some trial and error; however, the following guidelines may help.

General Rules for Choosing (u) and (dv)

9. Choose (u) so that $(u' = \frac{du}{dx})$ is simpler than (u)

   a (possibly maybe (u) will even disappear in the second integral because $(u' = 1)$

10. The second factor of the integrant must then be (dv).

    a You will need to turn (dv) into v by integration, so make sure that you can integrate (dv).

A good acronym to use in choosing (u) is

LAE

Logarithmic, Algebraic, Exponential

Since we are only covering these types of functions in Business Calculus, these are the only functions we need to concentrate on.

Choose (u) to be the function that appears first in this list.

- In other words, if a **logarithmic** function is one of the factors, make that your (u).
- If you do not have any logs, make the **algebraic** function factor (polynomials, radicals, rational) functions your (u).
- If you do not have any logs or algebraic functions, make the **exponential** function your (u).

Herbert Kasube's article on the LIATE method ("A technique for integration by parts") "American Mathematical Monthly, Vol.90, 1983

Additional Tips for Solving various Complex Integrals

Integrals of the Form	Select the proper (du), (dv) and (u) for the substitution integration	
$\int e^{ax}\,dx$	$u = ax$ $du = a\,dx$	
$\int x^n e^{ax}\,dx$	$dv = e^{ax}\,dx$	$u = x^n$
$\int (x+a)(x+b)^n\,dx$	$dv = (x+b)^n\,dx$	$u = (x+a)$
$\int x^n \ln(x)\,dx$	$dv = x^n\,dx$	$u = \ln(x)$
$\int (\ln(x))^n\,dx$	$dv = e^u\,du$	$u = \ln(x)$ $x = e^u$

1. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

2. $\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\cdot\int\left(x^{(n-1)}e^{ax}\right)dx$

3. $\int (x+a)(x+b)^n\,dx = \frac{(x+a)(x+b)^{(n+1)}}{(n+1)} - \frac{1}{(n+1)}\cdot\int (x+b)^{(n+1)}\,dx$

4. $\int x^n \ln(x)\,dx = \left[\frac{x^{(n+1)}}{(n+1)^2}\right]\left[(n+1)\cdot\ln(x) - 1\right] + C \quad ; \quad n \neq 1$

5. $\int (\ln(x))^n\,dx = x\cdot(\ln(x))^n - n\cdot\int\left[(\ln(x))^{(n-1)}\right]dx$

Example Problem #1

Finding indefinite integrals using Integration by Parts

Using the Integration by Parts, find the antiderivative, ($f(x) = xe^x$):

$$\int xe^x dx$$

Solution:

First find the four parts of integration: (u), (v), (du), (dv).

7. Find (u) and (du):

$$u = x \quad ; \quad du = dx$$

8. Find (dv) and (v):

$$dv = e^x dx$$

$$v = \int dv = \int e^x dx = e^x + K$$

9. Next substitute the four parts back into the Integration by Parts formula:

$$\int u\,dv = uv - \int v\,du$$

$$\int u\,dv = xe^x - \int e^x\,dx$$

$$\int u\,dv = e^x x - e^x + K$$

$$\int u\,dv = e^x(x - 1) + K$$

In the example, we could have chosen either (x) or (e^x) as our (u), but had we chosen $(u = e^x)$, the second integral would have become messier, rather than simpler.

Example Problem #2

Evaluate definite integrals using Integration by Parts

Using the Integration by Parts, evaluate the definite integral

$$\int_1^4 6x^2 ln(x)\, dx$$

Solution:

First find the four parts of integration: (u), (v), (du), (dv).

1. Find (u) and (du):

$$u = \ln(x) \quad ; \qquad du = \left(\frac{1}{x}\right) dx$$

2. Find (dv) and (v):

$$dv = 6x^2 dx$$

$$v = \int dv = 6\int x^2 dx = 2x^3 + K$$

3. Next substitute the four parts back into the Integration by Parts formula:

$$\int u\, dv = uv - \int v\, du$$

$$\int u\, dv = 2x^3 \ln(x) - \int 2x^3 \left(\frac{dx}{x}\right)$$

$$\int u\, dv = 2x^3 \ln(x) - 2\int x^2\, dx$$

$$\int u\, dv = 2x^3 \ln(x) - \frac{2}{3}x^3 + K$$

$$\int u\, dv = 2x^3 \left[\ln(x) - \frac{1}{3}\right] + K$$

Example Problem #2 – Cont'd

Evaluate using Integration by Parts

4. Next, include the limits of integration:

$$\int_1^4 u\,dv = \left[2x^3\left[\ln(x) - \frac{1}{3}\right]\right]_1^4$$

$$\int_1^4 u\,dv = \left[2(4)^3\left[\ln(4) - \frac{1}{3}\right]\right] - \left[2(1)^3\left[\ln(1) - \frac{1}{3}\right]\right]$$

$$\int_1^4 u\,dv = 128 \cdot \ln(4) - \frac{128}{3} + \frac{2}{3}$$

$$\int_1^4 u\,dv = 128 \cdot \ln(4) - \frac{126}{3}$$

$$\int_1^4 u\,dv = [128 \cdot \ln(4) - 42] = 135.4457$$

Example Problem #3

Using the Integration by Parts, find the antiderivative of

$(f(x) = (x - 4)(x + 3)^5)$:

$$\int (x - 4)(x + 3)^5 dx$$

Solution:

First find the four parts of integration: (u), (v), (du), (du).

1. Find (u) and (du):

$$u = (x - 4) \quad ; \quad du = dx$$

2. Find (dv) and (v):

$$dv = (x + 3)^5 dx$$

$$v = \int dv = \int (x + 3)^5 dx = \frac{(x + 3)^6}{6} + K$$

3. Next substitute the four parts back into the Integration by Parts formula:

$$\int u\, dv = uv - \int v\, du$$

$$\int u\, dv = \frac{(x - 4)(x + 3)^6}{6} - \frac{1}{6}\int (x + 3)^6\, dx$$

$$\int u\, dv = \frac{(x - 4)(x + 3)^6}{6} - \frac{1}{6}\left(\frac{(x + 3)^7}{7}\right) + K$$

$$\int u\, dv = \frac{(x + 3)^6}{6}\left[x - 4 - \frac{1}{7}(x + 3)\right] + K$$

$$\int u\, dv = \frac{(x + 3)^6}{42}[6x - 31] + K$$

Continuous Income Stream – Integration by Parts

Recall the Continuous Stream of Income discussion we had in section 5.7. The problems we were able to solve in that section only involved a constant yearly stream of income, such as, $C(t) = Constant = 1000$.

This function means that you are adding $1000 to the account each year.

Using the integration by parts method, we can now solve problems where the continuous stream of income can be different each year, say,

$$C(t) = 1000t.$$

In other words, in the first year, you add $1000 to the account, in the second year $2000, and so on.

As, long as, we can model the flow of income with a function, we can use a definite integral to calculate the present and future value of a continuous income stream.

The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up (a definite integral) using the technique of integration by parts.

Here are one more time the formulas we discussed in section 5.7.

Present Value of a Continuous Stream of Income or Continuous Money Flow

Suppose **Principle** (P) money can earn interest at an **Annual Interest Rate** of (r), compounded continuously.

Let ($C(t)$) be a **Continuous Income Stream** function (in dollars per year) that applies between year (0) and year (T).

The **Present Value** of a **Continuous Stream of Income**

- ($C(t)$) dollars per year,
- (t) is the number of years from now, for (T) years,
- Continuous Interest Rate (r)

$$Present\ Value = P_F(T) = \int_0^{T\ (years)} C(t)e^{-rt}\,dt$$

The **Future Value** ($P_F(T)$) of a **Continuous Stream of Income** ($C(T)$) dollars per year, can be computed by the ordinary compound interest formula.

$$Future\ Value = P_F(T) = \int_0^{T\ (years)} C(t)e^{rt}\,dt$$

For the cases when the ($C(t)$) is **not** a constant function the Continuous Compounding formula ($A = Pe^{rt}$) does not hold, so

$$P_F(T) \neq Pe^{rT}$$

Example Problem #4

For a Continuous Stream of Income – Determine its Present Value

A very wealthy Investment Banker generates income at the rate of ($C(t) = 6t$) million dollars per year, where (t) is the number of years from now.

Find the "**Present Value**" of this **continuous stream of income** for the next ten (10 years) at the continuous interest rate of (5%).

Solution:

$$\begin{pmatrix} Present \\ Value \end{pmatrix} = P_V(T) = \int_0^{T\ (years)} C(t)e^{-rt}\, dt$$

$$P_V(T) = \int_0^{10} 6te^{-0.05t}\, dt$$

Steps for using the Integration by Parts

First find the four parts of integration: (u), (v), (du), (dv).

1. Find (u) and (du):

$$u = 6t \quad ; \quad du = 6dt$$

2. Find (dv) and (v):

$$dv = e^{-0.05t}\, dt$$

$$v = \int dv = \int e^{-0.05t}\, dt = -20e^{-0.05t} + K$$

3. Next substitute the four parts back into the Integration by Parts formula:

$$\int u\, dv = uv - \int v\, du$$

$$\begin{pmatrix} Present \\ Value \end{pmatrix} = \int u\, dv = -120te^{-0.05t} - \int (-20e^{-0.05t})\,(6dt)$$

Example Problem #4 – Cont'd

$$P_V(T) = \int u\,dv = -120te^{-0.05t} + 120\int e^{-0.05t}\,dt$$

$$P_V(T) = \int u\,dv = -120te^{-0.05t} - 2400e^{-0.05t} + K$$

$$P_V(T) = \int u\,dv = -120e^{-0.05t}(t + 20) + K$$

4. Next, include the limits of integration:

$$\begin{pmatrix}Present\\ Value\end{pmatrix} = \int_0^{10} u\,dv = [-120e^{-0.05t}(t + 20)]_0^{10}$$

$$\int_0^{10} u\,dv = \left[-120e^{-0.05(10)}(10 + 20)\right] - \left[-120e^{-0.05(0)}(0 + 20)\right]$$

$$\int_0^{10} u\,dv = [-3600e^{-0.5} + 2400] = \left[-\frac{3600}{e^{0.5}} + 2400\right]$$

$$\begin{pmatrix}Present\\ Value\end{pmatrix} = \int_0^{10} u\,dv = -1200\left(\frac{3}{e^{0.5}} - 2\right) \cong 216.5$$

Therefore, the "Present Value" of the "Stream of Income" over 10 years is approximately $216 million.

This means that $216 million at 5% interest compounded continuously would generate a continuous stream ($C(t) = 6t$) million dollars for 10 years.

This method of generating income, is often used to determine the worth of a company or some other asset with financial value; because it gives the "Present Value" of a future income, over the 10 years.

Example Problem #5

A company is considering purchasing a new machine for its production floor. The machine costs $65,000. The company estimates that the additional income from the machine will be a constant $7000 for the first year, then will increase by $800 each year after that.

To buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income.

Money can earn 1.7% per year, compounded continuously.

Should the company buy the machine?

Solution:

We will assume that the income will come in continuously over the 8 years.

We will also assume that interest rates will remain constant over that 8-year time-period (T).

We are interested in the present value of the machine, which we will compare to its $65,000 price tag.

Let (t) be the time, in years since the purchase of the machine.

The income from the machine is different depending on the time.

From ($t = 0$) to ($t = 1$) (the first year), the income is constant $7000 per year.

From ($t = 1$) to ($t = 8$), the income is increasing by $800 each year; the income flow function F(t) will be

$$F(t) = 7000 + 800(t - 1) = 6200 + 800t$$

To find the present value, we will have to divide the integral into the two pieces, one for each of the functions:

$$\begin{pmatrix} Present \\ Value \end{pmatrix} = P_V(T) = \int_0^{T\ (years)} F(t)e^{-rt}\,dt$$

$$P_V(T) = \int_0^1 7000e^{-0.017t}\,dt + \int_1^8 (6200 + 800t)e^{-0.017t}\,dt$$

Example Problem #5 – Cont'd

Next simplifying the above equation, a bit more yields the following.

$$P_V(T) \;=\; P_V(T)_1 \;+\; P_V(T)_2 \;+\; P_V(T)_3$$

$$P_V(T) \;=\; \int_0^1 7000e^{-0.017t}\,dt \;+\; \int_1^8 6200e^{-0.017t}\,dt \;+\; \int_1^8 800(t\,e^{-0.017t})\,dt$$

The first two of the three parts above do not require integration by parts to solve, but the third part above does require integration by parts to solve.

Where the first two parts are solved below:

$$P_V(T)_1 \;+\; P_V(T)_2 \;=\; \int_0^1 7000e^{-0.017t}\,dt \;+\; \int_1^8 6200e^{-0.017t}\,dt$$

$$P_V(T)_1 \;+\; P_V(T)_2 \;=\; \left(\frac{7000}{-0.017}\right)[e^{-0.017t}]_0^1 \;+\; \left(\frac{6200}{-0.017}\right)[e^{-0.017t}]_1^8$$

$$P_V(T)_1 + P_V(T)_2 = \left(\frac{7000}{-0.017}\right)[e^{-0.017} - 1] + \left(\frac{6200}{-0.017}\right)\left[e^{-0.017(8)} - e^{-0.017}\right]$$

$$P_V(T)_1 \;+\; P_V(T)_2 \;=\; 6940.83 \;+\; 40227.44 \;=\; 47{,}168.27$$

Next using the third part of the Present Value equation, find the four parts of integration: (u), (v), (du), (du).

$$P_V(T)_3 \;=\; \int_1^8 800(t\,e^{-0.017t})\,dt \;=\; uv \;-\; \int v\,du$$

Example Problem #5 – Cont'd

1. Find (u) and (du):

$$u = 800t \quad ; \quad du = 800dt$$

2. Find (dv) and (v):

$$dv = e^{-0.017t}\, dt$$

$$v = \int dv = \int e^{-0.017t}\, dt = -\left(\frac{1}{0.017}\right) e^{-0.017t} + K$$

3. Next substitute the four parts back into the Integration by Parts formula:

$$P_V(T) = \binom{Present}{Value} = \int u\, dv = uv - \int v\, du$$

$$P_V(T)_3 = -\left(\frac{800}{0.017}\right) te^{-0.017t} - \int \left(-\left(\frac{1}{0.017}\right) e^{-0.017t}\right) (800dt)$$

$$P_V(T)_3 = -\left(\frac{800}{0.017}\right) te^{-0.017t} + \left(\frac{800}{0.017}\right) \int e^{-0.017t}\, dt$$

$$P_V(T)_3 = -\left(\frac{800}{0.017}\right) te^{-0.017t} - \left(\frac{800}{0.017}\right)\left(\frac{1}{0.017}\right) e^{-0.017t} + K$$

$$P_V(T)_3 = \int u\, dv = -\left(\frac{800}{0.017}\right) e^{-0.017t} \left(t + \left(\frac{1}{0.017}\right)\right) + K$$

4. Next, include the limits of integration:

$$P_V(T)_3 = \int_1^8 800(t\, e^{-0.017t})\, dt = uv - \int_1^8 v\, du$$

$$P_V(T)_3 = \int_1^8 u\, dv = \left[-\left(\frac{800}{0.017}\right) e^{-0.017t} \left(t + \left(\frac{1}{0.017}\right)\right)\right]_1^8$$

Example Problem #5 – Cont'd

$$P_V(T)_3 = -\left[\left(\frac{800}{0.017}\right)e^{-0.017(8)}\left(8 + \left(\frac{1}{0.017}\right)\right)\right] - \left[-800(58.82)e^{-0.017}\left(1 + \left(\frac{1}{0.017}\right)\right)\right]$$

$$P_V(T)_3 = \left[\frac{-\left(\frac{800}{0.017}\right)\left(8 + \left(\frac{1}{0.017}\right)\right)}{e^{0.136}}\right] - \left[\frac{-\left(\frac{800}{0.017}\right)e^{-0.017}\left(1 + \left(\frac{1}{0.017}\right)\right)}{e^{0.017}}\right]$$

$$P_V(T)_3 = -2{,}744{,}772.96 - [-2{,}767{,}770.59]$$

$$P_V(T)_3 = \int_1^8 800(t\, e^{-0.017t})\, dt \cong 22{,}997.64$$

Next, we must return to sum all the Present Values:

$$P_V(T) = P_V(T)_1 + P_V(T)_2 + P_V(T)_3$$

$$P_V(T) = 6{,}940.83 + 40{,}227.44 + 22{,}997.64$$

$$P_V(T) = \$70{,}165.92$$

Therefore, the "Present Value" of the "Stream of Income" over 8 years is approximately $70,166.

This means that $70,166, at 1.7% interest compounded continuously, over 8 years, would generate a continuous income stream ($F(t) = 6200 + 800t$) dollars.

The present value of ($70166), for the income stream, is greater than the cost of the machine ($65,000), so the company should buy the machine!

5.9 - EXERCISES

Use integration by parts to find each integral.

1.	$\int x^3 lnx\, dx$	2.	$\int 2xe^{4x}\, dx$
3.	$\int (x + 5)e^{3x}\, dx$	4.	$\int \sqrt[3]{y}\, lny\, dy$
5.	$\int \frac{ln\, x}{x^3}\, dx$	6.	$\int 3ze^{-0.75z}\, dz$
7.	$\int (y - 6)^4(y + 7)\, dy$	8.	$\int \frac{x}{\sqrt[3]{x + 1}} dx$
9.	$\int \frac{5z}{e^{5z}}\, dz$	10	$\int (lnx)^2\, dx$ (Hint: you will need to use integration by parts twice.)

Evaluate each definite integral using integration by parts.

11.	$\int_1^5 x^4 ln\left(\frac{x}{5}\right)\, dx$	12.	$\int_0^4 2xe^{2x}\, dx$
13.	$\int_0^{ln5} 25ye^y\, dy$	14.	$\int_0^4 t\,(t - 4)^3\, dt$

15.	If a company's Marginal Profit function is ($MP(t)$), is given in millions of dollars. Find the Profit function ($P(t)$). Hint: Evaluate the constant (K) so that profit is ($P(t) = 0$) at ($t = 0$). $$MP(t) = te^{-2t}$$
16.	A Business Owner, generates a continuous stream of income of ($9t$) million dollars per year, from his investment company, where (t) is the number of years that the company has been in operation. Find the present value and future value of this stream of income over the first 7 years at a continuous interest rate of 9%?
17.	Find the present value of a continuous income stream ($F(t)$). Where (t) is in years, and ($F(t)$) is in tens of thousands of dollars per year, for 10 years, if money can earn 2% annual interest, compounded continuously. $$F(t) = 20 + t$$

Solutions

94. $\frac{1}{4}x^4 \cdot ln(x) - \frac{1}{16}x^4 + K = \frac{1}{4}x^4 \cdot \left(ln(x) - \frac{1}{4}\right) + K$

95. $\frac{1}{2}xe^{4x} - \frac{1}{8}e^{4x} + K = \frac{1}{2}e^{4x}\left(x - \frac{1}{4}\right) + K$

96. $\frac{1}{3}(x + 5)e^{3x} - \frac{1}{9}e^{3x} + K = \frac{1}{3}e^{3x}\left(x + \frac{14}{3}\right) + K$

97. $\frac{3}{4}y^{\frac{4}{3}} \cdot ln(y) - \frac{9}{16}y^{\frac{4}{3}} + K = \frac{3}{4}y^{\frac{4}{3}}\left(ln(y) - \frac{3}{4}\right) + K$

98. $-\frac{ln(x)}{2x^2} - \frac{1}{4x^2} + K = -\frac{1}{2x^2}\left(ln(x) + \frac{1}{2}\right) + K$

99. $-4ze^{-0.75z} - \frac{16}{3}e^{-0.75z} + K = -4e^{-0.75z}\left(z + \frac{4}{3}\right) + K$

100. $\frac{1}{5}(y + 7)(y - 6)^5 - \frac{1}{30}(y - 6)^6 + K =$

$$= \frac{1}{5}(y - 6)^5\left(\frac{5}{6}y + 8\right) + K$$

101. $\frac{3}{2}x(x + 1)^{\frac{2}{3}} - \frac{9}{10}(x + 1)^{\frac{5}{3}} + K =$

$$= \frac{3}{2}(x + 1)^{\frac{2}{3}}\left(\frac{2}{5}x - \frac{3}{5}\right) + K$$

102. $-ze^{-5z} - \frac{1}{5}e^{-5z} + K = -e^{-5z}\left(z + \frac{1}{5}\right) + K$

103. $x(\ln(x))^2 - 2x\ln(x) + 2x + K$

104. $-\frac{1}{5}\ln(5) - \frac{3124}{25} = -124.64$

105. $\frac{7}{2}e^8 + \frac{1}{2} = \frac{1}{2}(7e^8 + 1) = 10{,}433.85$

106. $125\ln(5) - 100$

107. $-\frac{256}{5} = -51.2$

108. $P(t) = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}$

109. $\$146.52\ million\ and\ \$339.20\ million$

110. $\$225.0770\ ten\ thousand = \$2{,}250{,}770$